How Schwarzschild could have discovered and fixed the problem with his metric

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Abstract: Karl Schwarzschild's landmark paper revealed the first ever exact solution to Einstein's gravitational field equation. This was a major scientific achievement. However, his solution for the metric of a spherically symmetric spacetime is different than the one found in textbooks today—because Schwarzschild assumed, like in Newtonian gravity, that only one singularity could exist. Given the newness of Einstein's general theory of relativity and the existing paradigm of Newtonian gravity, his assumption was natural. While other authors have previously pointed out this shortcoming and corrected it, the contribution of this paper is to show how Schwarzschild might have discovered it himself. A simple geometric proof indicates how Schwarzschild, with a different assignment of a single constant of integration, would have arrived at the metric found in textbooks today.
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Résumé: L'article historique de Karl Schwarzschild révèle la toute première solution exacte de l'équation du champ gravitationnel d'Einstein. Ce fut une réalisation scientifique majeure. Cependant, sa solution pour la métrique d'un espace-temps à symétrie sphérique est différente de celle trouvée dans les manuels scolaires aujourd'hui – parce que Schwarzschild supposait, comme dans la gravité newtonienne, qu'une seule singularité pouvait exister. Compte tenu de la nouveauté de la théorie de la relativité générale d'Einstein et du paradigme existant de la gravité newtonienne, son hypothèse était logique. Alors que d'autres auteurs ont déjà souligné cette lacune et l'ont corrigée, la contribution de cet article est de montrer comme Schwarzschild aurait pu la découvrir lui-même. Une simple preuve géométrique indique comme, avec une affectation différente d'une seule constante d'intégration, Schwarzschild serait arrivé à la métrique trouvée dans les manuels scolaires aujourd'hui.

Key words: Schwarzschild; Gravity; Black Hole; Singularity; General Relativity; Einstein's Equation.

I. INTRODUCTION AND MOTIVATION

In November of 1915, Einstein was in the last throes of completing his general theory of relativity. He had just been able to show that his theory matched the non-Newtonian behavior exhibited by the orbit of Mercury. At that time, Karl Schwarzschild was both the director of the Astrophysical Observatory at Potsdam and a soldier serving in WWI. While on leave from his service in WWI, Schwarzschild attended one of Einstein's lectures[1] revealing his new theory. Einstein clearly impressed Schwarzschild who continued to digest his theory after returning to the eastern front. Only a few months later in early 1916, Schwarzschild startled Einstein with the first-ever exact solution to his gravitational field equation. He had reason to be surprised because Schwarzschild showed that a closed form solution could exist for Einstein's coupled set of ten non-linear second-order partial differential equations which describe gravity as curved spacetime.

Indeed, the majority of Schwarzschild's 1916 paper[2] is a masterful demonstration of physics and mathematics. However, by assuming that only one singularity existed, he misidentified the location of the origin of his coordinate system. A 1999 English translation[2] of Schwarzschild's paper created renewed interest in his contribution. (A usable but inferior Wikipedia translation[e)] is also accessible.) A historical perspective[3] discusses how alternative derivations by Hilbert, Droste, and others contributed to the form of the metric found in textbooks[4–6] today. Clearly, the physics community has moved beyond Schwarzschild's original solution and today celebrate the achievement with his name attached to the most common metric used for a non-spinning spherical gravitational source.

The contribution of this paper is to show how Schwarzschild might have discovered and fixed the problem with his metric. In Section II, we present the metric and the

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corresponding equations of motion for a particle as provided in his 1916 paper.[2] Section III shows how Schwarzschild could have suspected that there was a problem in his derived metric by examining the predicted behavior of freely falling particles. Section IV shows in a new and novel manner precisely the cause of the problem, and provides a simple and conclusive proof that Schwarzschild misidentified the location of the origin of his coordinate system and the point gravitational source. Also, as others have done, we show how Schwarzschild's solution could have matched the present-day solution by a simple change in the assignment of a constant of integration. Section V provides more historical background and concluding remarks. To make it easy for the reader, the nomenclature used throughout this paper matches Schwarzschild's 1916 paper,[2] even though some his nomenclature is different than what is broadly used by the physics community today.

II. THE METRIC AND EQUATIONS OF MOTION FROM SCHWARZSCHILD'S 1916 PAPER

To review the solutions to Einstein's field equation for a static spherically symmetric spacetime, the reader is referred to Weinberg's book.[7] In his book, Weinberg provides solutions to this problem in isotropic and harmonic coordinates in addition to the commonly found coordinates in today's textbooks.[4–6] In particular, the harmonic coordinate solution that Weinberg provides is the same metric structure as Schwarzschild's "rectangular coordinates" mentioned early in development of his paper.[2] The field equation solved by Weinberg is the commonly found[4–6] fully covariant form involving the vanished Ricci tensor.

One important part of history that complicated Schwarzschild's derivation and possibly contributed to the problem with his metric was the fact that he used an early version[8] of Einstein's field equation that was not fully covariant. This early version required the coordinate condition

$$g = -1, \tag{1}$$

where $g = \det(g_{\mu\nu})$ with $g_{\mu\nu}$ being the metric components. Because of this requirement, Schwarzschild cleverly employed a "trick" involving a transformation[2] to conform to Eq. (1). After using transformed coordinates and proceeding through challenging calculations, he arrived at his metric

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \frac{1}{\left(1 - \frac{\alpha}{R}\right)} dR^2 - R^2\left(d\vartheta^2 + \sin^2\vartheta d\phi^2\right), \tag{2}$$

where

$$R = (r^3 + \alpha^3)^{\frac{1}{3}} \tag{3}$$

over the domain

$$0 < r < \infty. \tag{4}$$

In Eqs. (2)–(4), s is proper time and the speed of light is assumed to be unity. By requiring equivalence with Newtonian theory far from the source, Schwarzschild's constant α is found to be $\alpha = 2G_N M$, where G_N is Newton's universal gravitation constant and M is the mass of the point source. r is the radial coordinate, ϑ and ϕ are the polar and azimuthal angles used to describe a spherical geometry, and t is the time coordinate. By setting $dr = d\vartheta = d\phi = 0$ in the metric, t can be shown to be the proper time as measured by a stationary observer far from the source. In his paper, Schwarzschild did not explicitly specify the domain of r given in Eq. (4). However, this was clearly his intent because he specifically assigned $r = 0$ to be where he thought the origin of the coordinate system and the point mass source are located. This assignment is evident in the discussion of his Eq. (13).[2] It will be shown that Schwarzschild mistakenly identified the origin of his coordinate system, resulting in a metric different from that found in textbooks today.

While at first glance Eq. (2) may look identical to the metric commonly found in textbooks,[4–6] it is not because the definition of Eq. (3) and the domain of r in Eq. (4). It is interesting to note that Schwarzschild never referred to R as a coordinate, calling it an "auxiliary quantity" in the discussion after his Eq. (13).[2] Clearly, one purpose of R defined in Eq. (3) is to reduce the mathematical complexity of the metric. If we substitute Eq. (3) into Eq. (2), we get

$$\begin{aligned} ds^2 = &\left[1 - \frac{\alpha}{(r^3 + \alpha^3)^{\frac{1}{3}}}\right] dt^2 \\ &- \frac{r^4}{(r^3 + \alpha^3)\left[(r^3 + \alpha^3)^{\frac{1}{3}} - \alpha\right]} dr^2 \\ &- (r^3 + \alpha^3)^{\frac{2}{3}}\left(d\vartheta^2 + \sin^2\vartheta d\phi^2\right). \end{aligned} \tag{5}$$

Equation (4) also defines the domain of r used in Eq. (5). In the mathematical development that follows, Eqs. (2)–(4) will be used interchangeably with Eqs. (4) and (5).

In his paper, Schwarzschild also provided the following equations of motion:

$$\begin{aligned} &\left(1 - \frac{\alpha}{R}\right)\left(\frac{dt}{ds}\right)^2 - \frac{1}{\left(1 - \frac{\alpha}{R}\right)}\left(\frac{dR}{ds}\right)^2 - R^2\left(\frac{d\phi}{ds}\right)^2 \\ &\quad = \text{const.} = h = 1, \end{aligned} \tag{6}$$

$$R^2 \frac{d\phi}{ds} = \text{const.} = c, \tag{7}$$

and

$$\left(1 - \frac{\alpha}{R}\right)\frac{dt}{ds} = \text{const.} = 1. \tag{8}$$

With $\vartheta = \pi/2$, these equations assume that the motion of a freely falling particle is in the equatorial plane. Equation (6) is easily arrived at by dividing Eq. (2) through by ds^2. As such, Schwarzschild's constant h must be 1. Physically, h is the square of the speed of light, not angular momentum as often used in today's nomenclature. Equations (7) and (8) are constants of motion derived using the Euler–Lagrange equations. The constant c in Eq. (7) is not to be confused with the speed of light. The constant of 1 in Eq. (8) applies only for a particle released from rest at infinity or moving well below the speed of light when far from the source. This constant of 1 is arrived at by letting $r \to \infty$ and setting $dt/ds = 1$ for the initially stationary particle.

III. CLUES THAT A PROBLEM EXISTED

It is only natural that Schwarzschild would have viewed the gravity described by his metric from the perspective of his knowledge of Newtonian gravity. Since Newton's gravitational potential has a single singularity at a point mass source, it was not unreasonable for him to assume that only one singularity could exist in his metric. In fact, Schwarzschild explicitly required only one singularity as indicated in his list of conditions immediately after his Eq. (9).[2]

By exploring the trajectories of freely falling particles, Schwarzschild could have gained a sense that general relativistic gravity is different than Newtonian gravity, particularly near the point source. He could have discovered that difference by scrutinizing Eqs. (2)–(8). He used those equations but could very well have been focused on showing that his exact solution agrees with Einstein's approximate calculation for the precession of Mercury's orbit in a weak field. Indeed, had he explored the entirety of his spacetime with these equations, he would have discovered some surprising results. For example, he could have verified that a stationary clock close to the source runs slower than a stationary clock far from the source.[6] This would not have been a surprise to Schwarzschild, because Einstein predicted this behavior in 1907.[9] Furthermore, Schwarzschild armed with his exact solution would have concluded that time stands still or freezes for a clock at $r = 0$, where he believed the source to be, when compared to a stationary clock far from the source. This would have been a startling result for Schwarzschild to discover, but again it was not inconsistent with Einstein's earlier prediction of gravitational time dilation. Also, he would have found that a radially falling particle decelerates and comes to rest as it approaches the source when the motion is expressed in terms of the r and t coordinates. This result says that it takes an infinite amount of time,[10] as measured by an observer far from the source, for the particle to reach $r = 0$. Concerned that such behavior is not at all found in Newton's theory, Schwarzschild might then have wondered what a stationary observer near the source would have measured the same particle's speed to be. Again, using Eqs. (2)–(8), Schwarzschild would have found such an observer to measure the speed of the same particle to approach the speed of light.[10] This would have probably been accepted as a reasonable result because it supported Einstein's assertion that the local spacetime must conform to special relativity[11] with the speed of a particle limited to be less than the speed of light. Up to this point in his analysis, Schwarzschild would have discovered some astonishing but explainable behavior, particularly near where he believed his point source is located. However, without having gone in that direction, some troubling behavior would emerge when one presses forward and further studies the behavior of freely falling particles near where he thought the source to be located.

Of immediate concern should have been Eq. (7). When a particle with a nonzero tangential velocity passes infinitesimally close to the origin of a polar coordinate system, the angular velocity must approach infinity. Referring to the illustrative example of Fig. 1, the reason that the angular velocity approaches infinity is as follows. As shown, the particle's path, as it passes the origin, makes a right triangle with sides x and y (the rectangular coordinates), and hypotenuse r and angular displacement ϕ (the corresponding polar coordinates). When passing the origin, ϕ sweeps nearly 180°. Over an instant of time ds, the particle's speed is $v = dy/ds = d(x \tan\phi)/ds$ or $v = x(d\phi/ds)\cos^{-2}\phi$, from which the angular rate is $d\phi/ds = (v/x)\cos^2\phi$. Just as it passes the origin, $\phi = 0$ and the distance r from the origin assumes its smallest value of x, during which it follows as x goes to zero that $d\phi/ds = v/r = v/x$ becomes infinite. This behavior is purely a result of geometry and applies equally well to the local geometry of curved spacetime.

However, a different behavior is predicted in Eq. (7), which is Eq. (16) in Schwarzschild's paper.[2] As previously stated, this equation is a constant of motion that results directly from the Euler–Lagrange equations with constant c being analogous to angular momentum. When combined with Eq. (3), Eq. (7) can be written as

$$\frac{d\phi}{ds} = \frac{c}{(r^3 + \alpha^3)^{\frac{2}{3}}}. \tag{9}$$

This equation can be interpreted as the angular displacement $d\phi$ swept during a short time period ds as measured by a clock attached to the falling particle. Considering particle trajectories with a nonzero value for c (recall constant c is not the speed of light) passing ever closer to where he believed the source and origin of his coordinate system to be located, Schwarzschild should have been troubled by the angular velocity not approaching infinity. In fact, as shown in Eq. (9), as $r \to 0$ the angular speed $|d\phi/ds| \to c/\alpha^2$, which is clearly a finite value for $\alpha \neq 0$. This behavior suggests an underlying problem with Schwarzschild's understanding of the geometry represented by his metric.

Schwarzschild could have found other concerning behavior with his metric by studying a particle dropped from

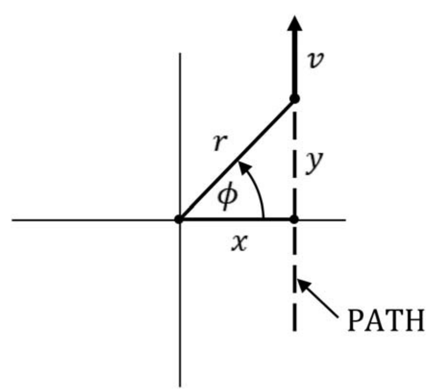

FIG. 1. Illustrative example of a particle passing near the origin.

rest at infinity. To fully explain how that concern arises, a brief review of Newtonian behavior is in order. In Newtonian gravity, a particle dropped from rest from a radial distance r_{Nd} from the point source has velocity[10]

$$\left(\frac{dr_N}{dt_N}\right)^2 = \alpha\left(\frac{1}{r_N} - \frac{1}{r_{Nd}}\right), \tag{10}$$

where r_N is the radial distance from the source that the particle reaches and t_N is Newton's universal time. Differentiating this equation with respect to t_N produces Newton's inverse square law. If the particle is dropped from $r_{Nd} = \infty$, Eq. (10) becomes

$$\frac{dr_N}{dt_N} = -\sqrt{\frac{\alpha}{r_N}}. \tag{11}$$

This equation can be separated and integrated. If we let $t_N = 0$ when $r_N = r_{No}$, then we get

$$t_N = \frac{2}{3}\left(\sqrt{\frac{r_{No}^3}{\alpha}} - \sqrt{\frac{r_N^3}{\alpha}}\right). \tag{12}$$

It is interesting that this formula is only valid for $r_N \geq 0$.

Schwarzschild could have considered the same scenario for his metric. By substituting Eq. (8) into Eq. (6) with $d\phi/ds = 0$, he would have obtained

$$\frac{dR}{ds} = -\sqrt{\frac{\alpha}{R}}. \tag{13}$$

Equation (13) resembles Eq. (11) and can be identically separated and integrated. Substituting Eq. (3) into the result of the integration, he would have derived

$$s = \frac{2}{3}\left(\sqrt{\frac{r_0^3 + \alpha^3}{\alpha}} - \sqrt{\frac{r^3 + \alpha^3}{\alpha}}\right), \tag{14}$$

where the interval of integration starts with proper time $s = 0$ as the falling particle passes through finite radial coordinate r_o. An interpretation of Eq. (14) is the time s of a clock attached to the particle as it falls through radial coordinate values. Schwarzschild would have noticed that s will continue to grow as the particle passes through $r = 0$, where he believed his point source to be located. In fact, Eq. (14) has no problem calculating the proper time of the particle as it passes through negative values of radial coordinate r until the particle arrives at $r = -\alpha$! This is shown in the graph of Fig. 2 where the proper time was set to $s = 0$ when the particle dropped from infinity passes through radial coordinate ratio $r/\alpha = 2$.

As shown in the graph, the particle passes $r = 0$ where Schwarzschild believed the point gravitational source to be located. A clock attached to the particle continues to accumulate proper time for the dashed portion of the trajectory until it reaches $r = -\alpha$. Clearly, the negative radial coordinate values of the falling particle, shown in dashed line, are part of its geodesic. Figure 2 and the analyses of Eqs. (9) and (14) provide compelling circumstantial evidence that there

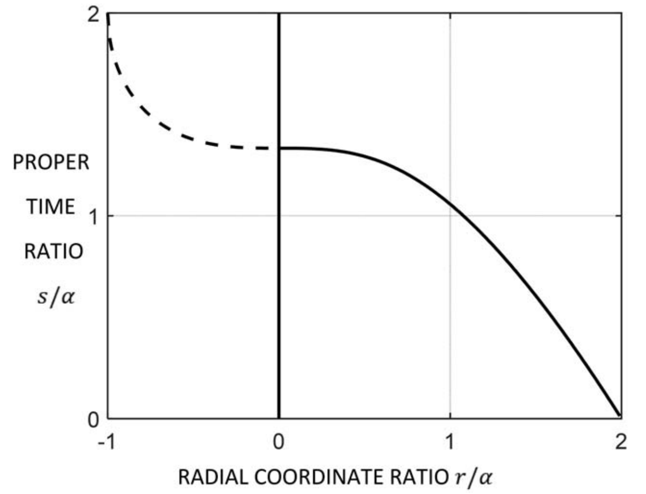

FIG. 2. Proper time versus the radial coordinate after the particle passes through $r/\alpha = 2$.

was a problem with Schwarzschild's identification of the origin of his coordinate system and location of his point source.

IV. IDENTIFYING AND FIXING THE PROBLEM

If Schwarzschild had done the study of freely falling particles in Section III, he would have been motivated to take another look at the geometry of his metric of Eqs. (2)–(4), or equivalently Eqs. (4) and (5). Due to the spherical symmetry, angles ϑ and ϕ have the same meaning in the flat space of Newtonian gravity as in the curved spacetime which Schwarzschild's metric was intended to describe. In the spacetime represented by Schwarzschild's metric, a curve with a constant radial coordinate in the equatorial plane is a *circle*. Using $\vartheta = \pi/2$, $dt = dr = d\vartheta = 0$ and setting $ds^2 = -d\sigma^2$ with σ being the proper spatial length, Schwarzschild would have reduced Eq. (5) to

$$d\sigma = (r^3 + \alpha^3)^{\frac{1}{3}} d\phi. \tag{15}$$

Integrating this with constant r over a complete revolution of ϕ, Schwarzschild would have derived

$$\sigma = (r^3 + \alpha^3)^{\frac{1}{3}} \int_0^{2\pi} d\phi = 2\pi(r^3 + \alpha^3)^{\frac{1}{3}} \tag{16}$$

for the circumference of a circle centered on the origin. This calculation only involves the geometry of Schwarzschild's spacetime and not the motion of a particle. The troubling result from Eq. (16) is that it indicates that a circle with a "radius" of radial coordinate $r = 0$ has a nonzero circumference $\sigma = 2\pi\alpha$! In fact, values of r for the range $-\alpha < r < 0$ produce positive circumferential length. Only when $r = -\alpha$ will $\sigma = 0$, yielding a circle with no circumference, i.e., a point. This *proves* that the origin of Schwarzschild's coordinate system is at $r = -\alpha$, despite the oddity of r being negative. Furthermore, since spacetime is spherically symmetric the source must be located there as well. This conclusion is at odds with Eq. (4) and indicates that the domain of Eq. (4) does not capture the entirety of the spherically symmetric spacetime. It is interesting to note that if he would have strictly adhered to the desired metric form[2] of his Eq. (6), Schwarzschild would have then avoided this issue entirely,[12] since he would have found that a circle would have had circumference $\sigma = 2\pi r\sqrt{G}$ where G was intended to be a yet-to-be-determined function of r. Equation (6) of his paper would have required the circumference of a circle to be zero when the radial coordinate is zero, which as we have just demonstrated is not the case for Eq. (5).

If the domain of r in Eq. (4) is ignored, Fig. 3 shows the corrected relationship between R and r in the equatorial plane of $\vartheta = \pi/2$.

As shown, the radial line A makes an angle ϕ. The coordinates r and R identify points along the radial line. We can see that $r = 0$ and $R = 0$ do not represent the same point. The circle defined by $r = 0$ and $R = \alpha$ has a circumference of $\sigma = 2\pi\alpha$ as required by Eqs. (3) and (16). The coordinate values $r = -\alpha$ and $R = 0$ define a *point*, that is, a circle having a circumference of $\sigma = 0$. This point is where the gravitational source must be located to create a spherically symmetric spacetime. The frozen clock and the falling particle decelerating to zero coordinate speed discussed at the beginning of Section III both occur at coordinate values $r = 0$ and $R = \alpha$. These behaviors are associated with the event horizon, a term[13] that was created many years after Schwarzschild's paper. Further, Fig. 3 resolves the conundrum associated with Eqs. (9) and (14). Letting $r \to -\alpha$ in Eq. (9) causes the angular velocity to approach infinity as was required. Enlarging the domain to include $-\alpha < r \le 0$ explains the additional proper time experienced by the radially falling particle predicted by Eq. (14). Also, Fig. 3 demonstrates that Eq. (3), which can be taken to be the transformation between the radial coordinates r and R, does not allow a simple translation of the origin from one point to another.

Armed with this new understanding, Schwarzschild could have interpreted the radial coordinate correctly by just assigning a different value to his integration constant ρ in his Eq. (12). Instead of assigning $\rho = \alpha^3$ in his Eq. (13)[2] based upon his belief that only one singularity could exist, he should have assigned $\rho = 0$ based upon the *geometry* of his coordinate system. This would have resulted in a metric as given by

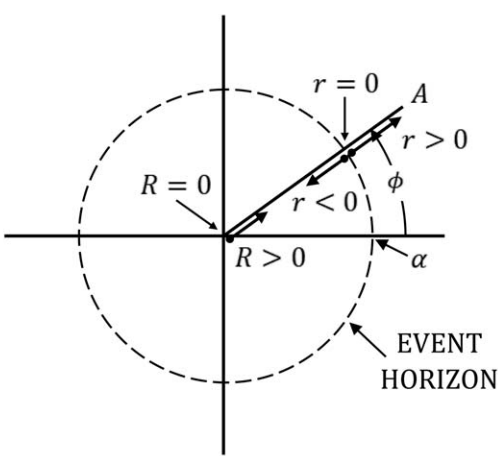

FIG. 3. The geometric relationship between r and R.

$$ds^2 = \left(1 - \frac{\alpha}{\hat{r}}\right)dt^2 - \frac{1}{\left(1 - \frac{\alpha}{\hat{r}}\right)}d\hat{r}^2 - \hat{r}^2\left(d\vartheta^2 + \sin^2\vartheta d\phi^2\right) \tag{17}$$

with new radial coordinate $\hat{r}$ having the domain

$$0 < \hat{r} < \infty. \tag{18}$$

This is the metric found in all textbooks[4–6] today. It includes two singularities at $\hat{r} = 0$ and at $\hat{r} = \alpha$. Note that $\hat{r}$ is used here to distinguish it from Schwarzschild's radial coordinate; however, modern textbooks simply use r, not to be confused with the radial coordinate in Schwarzschild's original paper.

V. HISTORICAL PERSPECTIVE AND CONCLUSIONS

Both Eqs. (2) and (5) are indeed solutions to Einstein's covariant field equation, which is Schwarzschild's great accomplishment. However, it was the misidentification of the origin of his coordinate system, and therefore the location of his point source, that results in a discrepancy between Schwarzschild's metric and the metric found in today's textbooks. This misidentification of the origin led to the truncated domain of Eq. (4) and the exclusion of the region directly adjacent to the point source. As shown in Sections III and IV, everything was present in Schwarzschild's paper to be able to correctly identify the origin of his coordinate system and the location of the point source.

The early version of Einstein's theory, which included the coordinate condition of Eq. (1), may have contributed to Schwarzschild embracing r as the true radial coordinate, leaving the auxiliary quantity R as only an intermediate calculation. However, Schwarzschild being an obviously gifted scientist and mathematician would have surely considered Eq. (3) to be a valid coordinate transformation. It is difficult to believe that he did not find it tempting to identify $R = 0$ as the origin of his polar coordinate system. It appears that Schwarzschild's overriding belief that only one singularity could exist must have led him to incorrectly identify the origin of his coordinate system. There is reason to suspect that Schwarzschild added condition number 4 following his Eq. (9) *after* his derivation in order to justify his identification of the location of the origin and point source. It would be years before the correct form of the metric in Eq. (17) with two singularities would be understood. The modern understanding[14] of Eqs. (17) and (18) is that a physical or geometric singularity exists at $\hat{r} = 0$, the location of the source, and that a coordinate singularity exists at $\hat{r} = \alpha$, the location of the event horizon. Additionally, it can be shown that the curvature is singular at the source but well behaved at the event horizon.[15] The concept of a blackhole, which Eq. (4) excludes, took decades[16] after Schwarzschild's paper to take root.

Some[17] have questioned whether Schwarzschild's name should be associated with the metric found in textbooks today, because his metric excluded the region $-\alpha < r \le 0$. On the other hand, others[18] may be overly forceful in defending Schwarzschild's legacy by asserting technical arguments that try to justify some of the shortcomings found in his 1916 paper. Science as it advances often does not get everything perfect from the start, particularly when revolutionary ideas are involved. Certainly, Einstein's own journey toward general relativity involved notable missteps.[8] It is with this true nature of discovery that we are more than comfortable with Schwarzschild's name being identified with the famous first exact solution to Einstein's field equation, even if all the details in his paper were not perfect.

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