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On a new field theory formulation and a space-time adjustment that predict the same precession of Mercury and the same bending of light as general relativity

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Abstract: This article introduces a new field theory formulation. The new field theory formulation recognizes vector continuity as a general principle and begins with a field that satisfies vector continuity equations. Next, independent of the new formulation, this article introduces a new space-time adjustment. Then, we solve the one-body gravitational problem by applying the space-time adjustment to the new field theory formulation. With the space-time adjustment, the new formulation predicts precisely the same precession of Mercury and the same bending of light as general relativity. The reader will find the validating calculations to be simple. The equations of motion that govern the orbital equations are in terms of Cartesian coordinates and time. An undergraduate college student, with direction, can perform the validations. © *2020 Physics Essays Publication*. [http://dx.doi.org/10.4006/0836-1398-33.4.489]

Résumé: Cet article présente une nouvelle formulation de la théorie des champs. Cette formulation de la théorie des champs reconnaît la continuité vectorielle comme un principe général et se base sur un champ qui satisfait les équations vectorielles de continuité. De plus, et indépendamment de cette nouvelle formulation, cet article introduit un nouvel ajustement de l'espace-temps. Enfin, nous résolvons le problème gravitationnel à un corps, en appliquant la modification de l'espace-temps à la nouvelle formulation de la théorie des champs. Avec cette modification de l'espace-temps, cette nouvelle formulation prédit précisément la même précession de Mercure et la même courbure de la lumière que la relativité générale. Le lecteur trouvera les calculs de validation simples. Les équations du mouvement qui gouvernent les équations orbitales sont fonction de coordonnées du repère cartésien et du temps. Un étudiant de premier cycle, avec un peu d'orientation, pourra effectuer les calculs de validation soi-même.

Key words: Newtonian Theory; Field Theory; General Relativity; Gravitation; Precession of Mercury; Bending of Light; Vector Continuity.

I. INTRODUCTION

One describes mathematically any space-time field that has flow lines that never begin, nor end, nor cross, as a four-dimensional vector function that satisfies vector continuity equations [see Appendix A for the development of the new field theory (FT) formulation]. The vector continuity equations are general equations that reduce to conservation laws, to wave equations, and to potential equations. Therefore, in retrospect, it was never a coincidence in Newtonian theory (NT) that the gravitational potential satisfies potential equations. It was never a coincidence in electromagnetic theory (EM) that Maxwell's equations describe fields that satisfy wave equations within a given frame of reference. Vector continuity appears throughout NT, EM, and special relativity (SR). One of the novelties of the new FT formulation is that it begins with vector continuity.

NT, EM, SR, and general relativity (GR) claim different territories in the landscape of physics, and their territories overlap. The formulations differ by their metrics, by their frames of reference, and by their coordinate systems. Scientists have attempted to connect one theory to the other. For example, as it pertains to the connection between NT and GR, Atkinson[1] examined GR in Euclidean terms and Montanus[2] developed a formulation of GR in so-called absolute Euclidean space-time. Sideris[3] asked fundamental questions about the connections between NT and GR, and Ziefle[4] modified NT by the introduction of "gravitons" in an attempt to predict the precession of Mercury and the bending of light. These and other researchers strengthened the belief that important connections exist between NT, EM, SR, and GR. However, the connections continue to be confusing, and the work is not complete.

To address this confusion, let us now distinguish between two classes of physical theories: The class-A theory addresses physical behavior within a frame of reference, and the class-B theory addresses physical behavior across two or more frames of reference. This article focuses on the class-A theory, in particular, the new FT formulation and on a new space-time adjustment that we apply to gravitation in the

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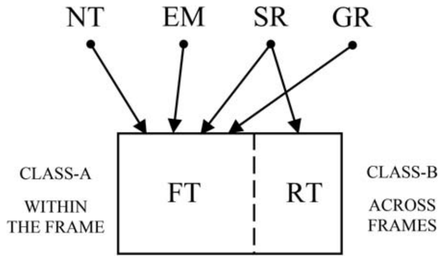

FIG. 1. The territories of analysis of FT and RT.

new FT formulation. We will refer to the class-B theory as relativity theory (RT). To be clear, by frame of reference, we mean the frame in which one takes measurements and makes observations. This is quite different from what one means by the coordinate system. One can employ different coordinate systems within a frame of reference, and one can employ the same coordinate system across different frames of reference at the instant the frames are coincident. One employs the Galilean transformation between coordinate systems within a frame of reference and the Lorentz transformation across frames of reference regardless of the coordinate system. As depicted in Fig. 1, NT, EM, and GR nicely fit in FT, and SR fits in both FT and RT.

NT and EM treat time as a parameter within a three-dimensional frame of reference. SR was the first theory to employ space-time within a four-dimensional frame of reference. It was also the first theory to address the differences in behavior across different four-dimensional frames of reference. Both of these advancements were as significant to physics as much as they were a source of confusion to those who did not appreciate the significance of the distinction. Like NT and EM, SR employed the inertial frame of reference. GR was the first theory to employ curved space-time and the first to make a number of predictions pertaining to gravity. GR was first to predict accurately the precession of orbiting bodies and the bending of light around massive bodies. GR focused on a metric-changing transformation between coordinate systems within a frame of reference, commonly described as between flat and curved space-times within the frame of reference. Relatively theory could include parts of GR depending on how one classifies GR. Ultimately, the distinction between a class-A theory and a class-B theory is important to one's understanding of the relationships among NT, EM, SR, and GR, and in understanding how they compare with the FT formulation developed in this article.

The Merriam-Webster dictionary defines analysis as the separation of a whole into its component parts. In this spirit, the new FT formulation begins by interpreting the whole as a field that blankets space-time and satisfies vector continuity equations. We refer to the whole as the 4D energy vector field or just as energy. We shall refer to the component parts of the whole as fragments of energy.

The body of this article focuses on validating FT with the space-time adjustment for gravitation. The validation consists of showing that the precession of Mercury and the bending of light predicted by FT with the space-time adjustment agree precisely with the corresponding predictions by GR. The comparisons with GR are based on the solution to Einstein's field equation known as the Schwarzschild solution—the solution that describes the gravitational field that surrounds a spherical mass, on the assumption that its electric charge, angular momentum, and universal cosmological constant are all zero. The solution is a useful approximation applicable to astronomical objects such as many stars and planets, including the earth and the sun. The reader will find the new orbital equations and the steps involved in the validations to be simple. The orbital equations are quite simple to program, and curvilinear coordinates are not involved. An undergraduate college student, with direction, can perform the validations.

Appendix A contains the development of the new FT formulation. Appendix B sets up the one-body gravitational problem and other mathematical details needed to solve the one-body gravitational problem by the new FT formulation. Appendices C–G explain other properties of the FT formulation. Appendix H derives the effective radial potential in GR from the Schwarzschild solution. Finally, Appendix I reviews four-dimensional field theory.

II. INTRODUCTION TO THE NEW FT FORMULATION

Before proceeding to the validations, we briefly introduce the new FT formulation and compare it with the present-day formulation. The present day formulation starts with a relativistic correction in inertial space of the law of inertia in NT; the present-day field theory is a relativistic NT. It replaces the law $F_r = m\frac{d}{dt}v_r \ \ (r = 1, 2, 3)$, which describes the interaction force vector responsible for changing the state of a particle, with the relativistic law $F_r = m\frac{d}{dt}\left(v_r/\sqrt{1-(v/c)^2}\right)(r = 1, 2, 3)$. Notice in the law of inertia, with and without the relativistic correction, that one obtains the force vector by time differentiation. Furthermore, notice that the relativistic velocity components $v_r/\sqrt{1-(v/c)^2} \ \ (r = 1,\ 2,\ 3)$ align with the first three components of a 4D unit vector that is tangent to the particle's space-time path. The authors characterize the present-day field theory as *intending* to accommodate the transition from the particle to the field and from the spatial domain to the space-time domain. We now contrast the present-day formulation with the new formulation.

The new FT formulation begins differently. It replaces the spatial domain with the space-time domain. The kinematic quantities in the spatial domain differ from the corresponding quantities in the space-time domain. In the spatial domain, one employs position vector components $x_r \ (r = 1,\ 2,\ 3)$ of particles, 3D velocity vector components $v_r \ (r = 1,\ 2,\ 3)$ (time derivatives of x_r), and 3D acceleration vector components $a_r \ (r = 1,\ 2,\ 3)$ (time derivatives of v_r). In contrast, in the space-time domain, the components of a field point are $x_r \ (r = 1,\ 2,\ 3,\ 4)$. One employs 4D position vector components $x_{0r} \ (r = 1,\ 2,\ 3,\ 4)$ of source points, unit vector components $e_r \ (r = 1,\ 2,\ 3,\ 4)$ (path derivatives of x_{0r}), and 4D curvature vector components

$k_r\,(r = 1,\ 2,\ 3,\ 4)$ (path derivatives of e_r). The path derivative is a derivative with respect to the path increment $ds = \sqrt{dx_{0r}dx_{0r}}$ (repeated indices are summed) (Appendix I). Note that the term curvature that one uses in space-time geometry and that one uses in GR are different. With the space-time representation of kinematic quantities in place, the next step is to drape a 4D energy vector field over the space-time domain. The energy vector has field lines that never begin, nor end, nor cross, which one expresses mathematically by 4D vector continuity equations. The vector continuity equations reduce to wave equations (Appendix G) and to conservation laws and potential equations (Appendix I). Furthermore, 4D vector continuity motivates the functional form of the fragment of energy, from which one builds up the 4D energy vector $A_r\,(r = 1,\ 2,\ 3,\ 4)$ (Appendix D). The form of the simple fragment of energy is $A_r = -\frac{\alpha}{r}e_r\,(r = 1,\ 2,\ 3,\ 4)$, where α denotes the intensity of the fragment and where $r^2 = (x_r - x_{0r})(x_r - x_{0r})$ (repeated indices are summed from 1 to 3). With this new foundation, the next step is to define the action force vector and the interaction force vector. The components of the action force vector by one fragment or the resultant action force vector by a set of fragments are $P_r\ (r = 1,\ 2,\ 3,\ 4)$ (path derivatives of A_r), and the components of the 4D interaction force vector by one fragment acting on another fragment or the resultant interaction force vector acting on another fragment are $F_r = -\alpha P_r\ \ (r = 1,\ 2,\ 3,\ 4)$, where again α is the intensity of the fragment being acted on. Next, one recognizes a unique property of path differentiation in a space-time domain; that the path derivative of any continuous 4D vector field (in our case, the 4D energy vector) partitions mathematically into a conservative part and a Lorentzian part. The conservative part corresponds to mechanical action and the Lorentzian part to electromagnetic action (Appendix E). Finally, in the new FT formulation, we employ path differentiation to set up the change equation $k_r = P_r\ \ (r = 1,\ 2,\ 3,\ 4)$, where k_r are the components of the curvature vector of a fragment of energy, and where P_r are the components of the resultant action force vector that acts on the fragment of energy (Appendix A). With respect to the two formulations, first, note that the change equation maps to the relativistic law of inertia. One finds that the change equation and the relativistic law of inertia map one-to-one by comparing the time derivatives of the relativistic linear momentum components with the force components in the new FTformulation. The force components F_{Er} and F_{Nr}, corresponding to the present-day formulation and the new formulation, map one-to-one when $F_{Er} = F_{Nr}\sqrt{1 - (\frac{v}{c})^2}$. Next, note that the new formulation replaces the particle and the wave with the field along with its fragment of energy, and introduces a connection between mechanics and electromagnetism through path differentiation.

III. THREE FORMULATIONS

The fundamental problem of gravitation is a two-body problem. Whether employing FT, NT, or GR, one converts the two-body problem into a one-body problem (Appendix B). The following compares the treatments of the one-body problem in FT and NT with its treatment in GR. As will be seen below, the treatments require a space-time adjustment. Table I lists the fragment of energy, energy, action force, and interaction force for the one-body gravitational problem and more generally the governing equations (change equations and laws of inertia). For the one-body gravitational problem, the orbital mechanics takes place in the x_1, x_2 plane, the mass of the sun is M, and the mass of the orbiting body is m, where the orbiting body is either Mercury or a photon.

In general, GR starts with general covariance principles by which one determines a class of transformations between 4D Euclidean and 4D non-Euclidean (curved) space-times. In the absence of electromagnetic effects, the force vector in non-Euclidean space-time is zero; the particle follows a geodesic. Note that we will not review here the formidable apparatus of curved space-times. However, with this apparatus, one obtains a metric for the gravitational problem called the Schwarzschild metric. The Schwarzschild metric leads to the so-called Schwarzschild solution. The Schwarzschild solution can be manipulated to reveal an effective radial potential[b),5] and a corresponding interaction force vector in 4D Euclidean space-time, notwithstanding that the mathematical derivation lacks some reasoning[6] (see Appendix H for the development of the effective radial potential from the Schwarzschild solution). Finally, one determines a 3D acceleration vector from the interaction force vector.

A. Gravitation in the new FT formulation with no space-time adjustment

In the one-body gravitational problem, the field consists of just one fragment of energy. TABLE Ia gives its components. The term A given there is the magnitude of the vector field with components $A_s\ (s = 1,\ 2,\ 3,\ 4)$. Next, an action force is determined. Then, one determines 4D curvature vector components $k_s\ (s = 1,\ 2,\ 3,\ 4)$ by substituting the action force vector into the change equation.

B. Gravitation in the new FT formulation with a space-time adjustment

Table Ib gives the components $A_s\ (s = 1,\ 2,\ 3,\ 4)$ of the vector field of a stationary fragment using a space-time adjustment that we introduce for gravitation. We first apply it here to the new FT formulation and later to NT. The space-time adjustment adjusts the relationship between the speed of the source point of the orbiting fragment upon which the field of a stationary fragment acts, such as Mercury or a photon, and the orbiting fragment's corresponding angular momentum. Start by considering the circular motion of an orbiting fragment. In FT, the following expression is exact for circular motion:

[b)]Wikipedia, Two-body problem in general relativity, [Online]. Available: https://en.wikipedia.org/wiki/Two-body_problem_in_general_relativity

TABLE I. Gravitation in FT and NT.

a) Gravitation in the new FTformulation with no space-time adjustment (Appendices A and B)	
Fragment of energy	$A_1 = 0,\ A_2 = 0,\ A_3 = 0,\ A_4 = -\frac{Mc^2}{r}$
Energy	$A = -\frac{Mc^2}{r}$
Action force	$P_1 = \frac{GM}{c^2}\frac{x_1}{r^3},\ P_2 = \frac{GM}{c^2}\frac{x_2}{r^3},\ P_3 = 0,\ P_4 = 0$
Change equations	$P_1 = k_1,\ P_2 = k_2,\ P_3 = k_3,\ P_4 = k_4$
Interaction force	$F_1 = -mc^2P_1,\ F_2 = -mc^2P_2,\ F_3 = -mc^2P_3,\ F_4 = -mc^2P_4$
b) Gravitation in the new FT formulation with the space-time adjustment (Appendices A and B)	
Fragment of energy	$A_1 = 0,\ A_2 = 0,\ A_3 = 0,\ A_4 = -\frac{Mc^2}{r}\ \frac{1}{1-\left(\frac{v}{c}\right)^2}$
Energy	$A = -\frac{Mc^2}{r} - Mc^2\left(\frac{h}{mc}\right)^2\frac{1}{r^3},\ h = rmv\beta,\ \beta = \left(1-\left(\frac{v}{c}\right)^2\right)^{-1/2}$
Action force	$P_1 = \frac{GM}{c^2}\frac{x_1}{r^3} + 3\frac{GM}{c^2}\left(\frac{h}{mc}\right)^2\frac{x_1}{r^5},\ P_2 = \frac{GM}{c^2}\frac{x_2}{r^3} + 3\frac{GM}{c^2}\left(\frac{h}{mc}\right)^2\frac{x_2}{r^5}\,P_3 = P_4 = 0$
Change equations	$P_1 = k_1,\ P_2 = k_2,\ P_3 = k_3,\ P_4 = k_4$
Interaction force	$F_1 = -mc^2P_1,\ F_2 = -mc^2P_2,\ F_3 = -mc^2P_3,\ F_4 = -mc^2P_4$
c) Gravitation in NT with no space-time adjustment[a]	
Potential energy	$V = -\frac{GMm}{r}$
Interaction force	$F_1 = -\left(\frac{GMm}{r^3}\right)x_1,\ F_2 = -\left(\frac{GMm}{r^3}\right)x_2,\ F_3 = 0$
Law of inertia	$F_1 = ma_1,\ F_2 = ma_2,\ F_3 = ma_3$
d) Gravitation in NT with the space-time adjustment [or by GR,[b] (Appendix H)]	
Fragment of energy	$V_1 = 0,\ V_2 = 0,\ V_3 = 0,\ V_4 = -\frac{GMm}{r}\left(1+\left(\frac{v}{c}\right)^2\right)$
Potential energy	$V = -\frac{GMm}{r} - \frac{GMH^2}{c^2mr^3},\ H = rmv$
Interaction force	$F_1 = -\left(\frac{GMm}{r^3} + \frac{3GMH^2}{c^2mr^5}\right)x_1,\ F_2 = -\left(\frac{GMm}{r^3} + \frac{3GMH^2}{c^2mr^5}\right)x_2,\ F_3 = 0$
law of inertia	$F_1 = ma_1,\ F_2 = ma_2,\ F_3 = ma_3$

[a]Wikipedia, "Newton's law of universal gravitation," [Online]. Available at https://en.wikipedia.org/wiki/Newton%27s_law_of_universal_gravitation
[b]Wikipedia, "Two-body problem in general relativity," [Online]. Available at https://en.wikipedia.org/wiki/Two-body_problem_in_general_relativity

$$\frac{1}{1-\left(\frac{v}{c}\right)^2} = 1 + \left(\frac{h}{mc}\right)^2\frac{1}{r^2},\ \text{where } h = \frac{mrv}{\sqrt{1-\left(\frac{v}{c}\right)^2}}, \tag{1}$$

where h is angular momentum, m is mass, and r is the distance to the source point. To verify that this expression is exact, substitute h into the right side of the expression and then combine the two terms. The space-time adjustment consists of first multiplying the magnitude $-Mc^2\frac{1}{r}$ of the stationary fragment in Table Ia by the left side of the first expression in Eq. (1). Given that v is constant, one preserves vector continuity (see A_4 in Table Ib). In the gravitational problem, the velocity v of the orbiting fragment is *not* constant, whereas the angular momentum of the orbiting fragment *is* constant. When one multiplies the magnitude $-Mc^2\frac{1}{r}$ of the stationary fragment by the right side of the first expression in Eq. (1), one finds that the mathematical expression of the stationary fragment splits into two parts, with h being constant (see A in Table Ib). The second step of the space-time adjustment is to replace the left side of Eq. (1) with the right side of Eq. (1). This eliminates the dependence on v from the expression for A, leaving the expression in terms of its independent spatial coordinates alone. Using this expression, we determine the action force at any point along the trajectory wherein the motion is *not* circular

Next, we consider that the sun and Mercury are not merely simple fragments of energy but that each is a bundle of fragments. The path derivative of the fragments of energy of each bundle has a mechanical part and an electromagnetic part [see Eq. (A6)]. We assume within each bundle that the electromagnetic part cancels. The components of the action force acting on the orbiting fragment (Mercury or a photon) are $P_r = S_r = \partial A/\partial x_r,\ (r = 1,\ 2,\ 3,\ 4)$. Holding h constant, while taking the derivatives, we get P_r ($r = 1, 2, 3, 4$) in Table Ib. Next, we consider the interaction force on the orbiting fragment by the action of the sun. From Eq. (A10), we get F_r ($r = 1, 2, 3, 4$) in Table Ib. Note that we no longer guarantee vector continuity to be satisfied after making the adjustment. One determines the components of a 4D curvature vector k_s ($s = 1, 2, 3, 4$) of the orbiting fragment by substituting the mechanical part of its action into the change equation in Table Ib. This yields orbital equations that we will solve numerically. The reader will discover that the use of the second term on the right side of Eq. (1), which is responsible for the adjustment, is also responsible for predicting the same precession of Mercury and bending of light as in GR.

C. Gravitation in NT with no space-time adjustment

Referring to Table Ic, NT with no space-time adjustment begins with a potential energy function V or with interaction force vector components F_s ($s = 1, 2, 3$). A gradient vector relates the two by $F_s = -\partial V/\partial x_s\ \ (s = 1,\ 2,\ 3)$. Then, one determines 3D acceleration vector components a_s ($s = 1, 2, 3$) by substituting the interaction force vector into the governing law of inertia. Note that the potential energy function in NT with no space-time adjustment is *not* associated with a potential energy vector function. The potential energy function for gravitation will recognize that V is associated with a potential energy vector function.

D. Gravitation in NT with a space-time adjustment

We begin with NT and now associate the potential energy V of a body with a potential energy vector function like in the new FT formulation. Next, we apply the space-time adjustment to the one-body gravitational problem in NT. Again, the space-time adjustment considers the relationship between the speed of the mass center of the orbiting body and its corresponding angular momentum and, again, we start by considering the circular motion of the orbiting body. In NT, the following expression is exact for circular motion:

$$1 + \left(\frac{v}{c}\right)^2 = 1 + \left(\frac{H}{mc}\right)^2 \frac{1}{r^2} \quad \text{where } H = rmv, \tag{2}$$

where H is the angular momentum of the orbiting body. The space-time adjustment consists of first multiplying the gravitational potential $-GM\frac{1}{r}$ of the stationary body in Table Ic by the left side of the first expression in Eq. (2). Given that v is constant, one preserves vector continuity (see V_4 in Table Id). Again, in the one-body gravitational problem, the speed v of the orbiting body is *not* constant whereas its angular momentum is constant. Again, when one multiplies the gravitational potential $-GM\frac{1}{r}$ by the right side of the first expression in Eq. (2), one finds that the expression splits into two parts, with H being constant (see V in Table I). Like in FT, the second step is to replace the left side of Eq. (2) with the right side of Eq. (2), which eliminates the dependence on v and expresses V in terms of its independent spatial coordinates alone. We determine the interaction force by the sun from $F_r = -\partial V/\partial x_r,\ (r = 1,\ 2,\ 3)$, holding H constant. Again, we used the resulting expression to determine the interaction force at any point along the trajectory wherein the motion is *not* circular (see F_r ($r = 1, 2, 3, 4$) in Table Id). Again, we no longer guarantee vector continuity to be satisfied after making the adjustment.

In this article, the reader will discover, had Newton added the previously mentioned second term in Eq. (2) to the gravitational potential, that he would have predicted the same precession of Mercury as in GR. On the other hand, he would not have been inclined to pursue that because during the time of Newton scientists had not yet observed the precession of Mercury, they had not yet developed the concept of vector continuity, nor had scientists yet developed the apparatus of curved space-times.

IV. RESULTS

This section solves two one-body gravitational problems—the precession of Mercury in its orbit around the sun and the bending of light grazing the surface of the sun—each by FT and by NT with and without the adjustments. Table II lists the physical constants used in both problems and Table III lists the accelerations used to compute the dynamic responses.

A body orbits the sun in the problem of the precession of Mercury and in the problem of the bending of light. In the bending of light problem, the body is a massless point traveling at the speed of light. One calls such a point a photon. In

TABLE II. Physical constants.

	Sun[a]	
M	Mass of the sun	1.989×10^{30} kg
r_s	Radius of the sun	696 000 000 m
	Mercury[b,c]	
m	Mass of mercury	3.3×10^{23} kg
r_p	Perihelion radius of Mercury	4.6×10^{10} m
r_b	Aphelion radius of Mercury	6.982×10^{10} m
A	Semimajor axis of Mercury	57.91×10^{9} m
e	Eccentricity of the orbit	0.20566
v_p	Perihelion velocity	58.98×10^{3} m/s
T	Orbital period	87.969 Earth days = 7 600 530 s
	Other[d,e]	
G	Gravitational constant	6.674×10^{-11} m^3/kg-s^2
c	Speed of light	2.99×10^{8} m/s

[a]Wikipedia, Sun, [Online]. Available: https://en.wikipedia.org/wiki/Sun
[b]Wikipedia, Mercury (planet), [Online]. Available: https://en.wikipedia.org/wiki/Mercury_(planet)
[c]David R. Williams, NASA Goddard Space Flight Center, https://nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfact.html
[d]Wikipedia, Gravitational constant, [Online]. Available: https://en.wikipedia.org/wiki/Gravitational_constant
[e]Wikipedia, Speed of light, [Online]. Available: https://en.wikipedia.org/wiki/Speed_of_light

TABLE III. Calculating the dynamic response.

a) New FT formulation without the adjustment

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\frac{m}{\mu}\left(1-\left(\frac{v}{c}\right)^2\right)\begin{bmatrix} 1-\left(\frac{v_1}{c}\right)^2 & -\left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) \\ -\left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) & 1-\left(\frac{v_2}{c}\right)^2 \end{bmatrix}\begin{pmatrix} \frac{GM}{r^2}\frac{x_1}{r} \\ \frac{GM}{r^2}\frac{x_2}{r} \end{pmatrix}$$

$$\mu = \frac{mM}{m+M}$$

b) New FT formulation with the adjustment

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\frac{m}{\mu}\left(1-\left(\frac{v}{c}\right)^2\right)\begin{bmatrix} 1-\left(\frac{v_1}{c}\right)^2 & -\left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) \\ -\left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) & 1-\left(\frac{v_2}{c}\right)^2 \end{bmatrix}\begin{pmatrix} \frac{GM}{r^2}\left(1+3\left(\frac{h}{mc}\right)^2\frac{1}{r^2}\right)\frac{x_1}{r} \\ \frac{GM}{r^2}\left(1+3\left(\frac{h}{mc}\right)^2\frac{1}{r^2}\right)\frac{x_2}{r} \end{pmatrix}$$

$$h = rmv\beta$$

c) NT without the adjustment

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\frac{m}{\mu}\frac{GM}{r^3}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

d) NT with the adjustment

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\frac{m}{\mu}\left(\frac{GM}{r^3}+\frac{3GMH^2}{c^2m^2r^5}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$H = rmv$$

each problem, we determined the trajectories by numerical integration of the accelerations of the orbiting body. We adopted a numerical approach instead of an analytical approach, because analytical results do not yet exist for all of the cases considered and because the numerical approach simplifies the verification of the results. We used the ODE solver DVERK78 within MAPLESOFT™.[c)] The number of digits of accuracy was set to Digits = 20. DVERK78 finds a numerical solution of state equations using a seventh-eighth order continuous Runge-Kutta method.

Next, consider the initial conditions corresponding to all of the cases in Table III. The initial angular momentum in FT is $h = mr_p v_p/\sqrt{1-(v_p/c)^2}$. (The subscript p stands for perihelion.) In NT, the initial angular momentum is $H = r_p m v_p$. In all cases, the parameters in all cases are calculated from the data in Table II. Also $a_1 = \dot{v}_1$, $a_2 = \dot{v}_2$, $v_1 = \dot{x}_1$, $v_2 = \dot{x}_2$, and $v = \sqrt{v_1^2 + v_2^2}$.

A. Precession of Mercury

GR predicted successfully the so-called anomalous precession of Mercury, over and above the precession caused by the other planets and solar oblateness. From GR, the analytically determined precession of Mercury is[d),e)]

$$\begin{aligned}\delta\varphi &= \frac{6\pi G(M+m)}{c^2 A(1-e^2)}, \\ &= 5.047\times 10^{-7}\ \mathrm{rad/orbit} \\ &= 0.104093\ \mathrm{arc-sec/orbit} \\ &= 43.2\ \mathrm{arc-sec/century}.\end{aligned}$$

Again, the numerically integrated trajectories began at Mercury's perihelion with the initial conditions $x_1 = r_p$, $x_2 = 0$, $v_1 = 0, v_2 = v_p$ and continued to its next perihelion, after a little more than a full revolution (see Fig. 2). In order to determine the precise location of the next perihelion, the time derivative of the orbit radius $dr/dt = (x_1\dot{x}_1 + x_2\dot{x}_2)/r$ was monitored, and the time when its value crossed zero was determined. At that instant, the predicted angle of precession was calculated numerically from $\delta\varphi = x_2/r_p$. Table IV shows the precession angles numerically determined by FT and NT with and without the adjustments. The values

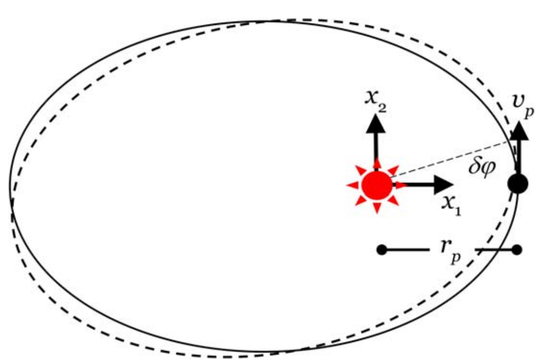

FIG. 2. (Color online) Trajectory of Mercury.

[c)]MAPLESOFT, [Online]. Available: https://www.maplesoft.com/
[d)]Wikipedia, "Two-body problem in general relativity," [Online]. Available: https://en.wikipedia.org/wiki/Two-body_problem_in_general_relativity
[e)]Wikipedia, "Schwarzschild geodesics," [Online]. Available: https://en.wikipedia.org/wiki/Schwarzschild_geodesics

TABLE IV. Angle of precession for each formulation.

Formulation	Precession angle	
	rad/orbit	arc-sec/century
a) FT without the adjustment	0	0
b) FT with the adjustment	5.047×10^{-7}	43.2
c) NT without the adjustment	0	0
d) NT with the adjustment	5.047×10^{-7}	43.2

predicted numerically by FT and NT with the adjustments agree to three decimal places with the celebrated analytical result from GR. Without any adjustment, FT and NT predict a precession angle of zero.

Figure 3 shows the graph of the precession angle of Mercury's orbit calculated by FT with the adjustment. It is shown as a function of dr/dt in the neighborhood of the completion of the first orbit, and it shows the zero crossing at 5.047×10^{-7} rad. We obtained the same result by NT with the adjustment; the two graphs overlap. Note that we also directly coded the Schwarzschild geodesic equations[f)] to determine the orbit using the same numerical method. The geodesic equations also led to a precession angle of 5.047×10^{-7} rad. This served as a numerical check, because one derives the GR effective potential from the Schwarzschild geodesic equations (Appendix H).

B. The bending of light

The classic problem of analytically determining the bending of light by GR predicts the angle of bending as[7,g)]

$$\delta_N = \frac{4GM}{c^2 r_p} = 8.534 \times 10^{-6}\,\text{rad} = 1.760\,\text{arc} - \text{sec},$$

where r_p is the "distance of closest approach" (the perihelion) and, for the purposes of simulation, has been set equal to the radius r_s of the sun (see Fig. 4). In order to predict the path of a photon orbiting the sun, we considered the model of a photon using the equations of motion in Table III. First, we set $\mu = mM/(m+M)$ to m after which the m cancelled from the equations of motion. In FT with the adjustment, by setting the photon speed to the speed of light, one would get a divide by zero error. The divide by zero arises in the term $(h/mc)^2 = (r_s v_2/c)^2 \frac{1}{1-(v/c)^2}$, when v is set equal to c. The expressions for the acceleration components become indeterminate (0/0), so we take the limit as v approaches c instead of evaluating v at c. One can also evaluate v at c by canceling the two $1-(v/c)^2$ terms to obtain $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{3GM}{r^5}\left(x_1 \frac{v_2}{c} - x_2 \frac{v_1}{c}\right)^3 \begin{pmatrix} -v_2/c \\ v_1/c \end{pmatrix}$ for the expression of the

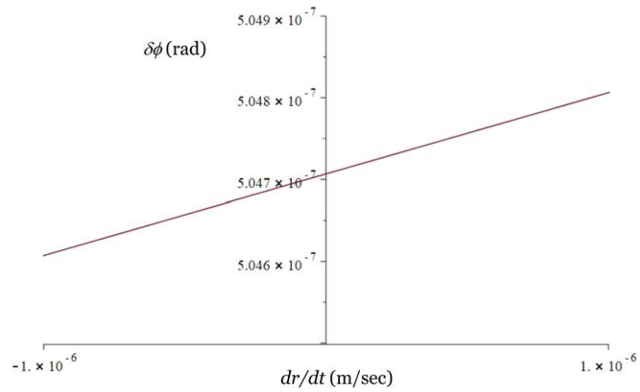

FIG. 3. (Color online) The precession angle in FT with the adjustment.

acceleration components of a photon under the action of a stationary source of mass M. We determined the angle of light bending by letting the photon speed approach the speed of light in a limiting process. In NT, we numerically integrated the photon trajectory starting at the Sun's radius with a velocity approaching the speed of light using $x_1 = r_s, x_2 = 0,\ v_1 = 0, v_2 \cong c$ as initial conditions and continued forward in time. The angular momentum term $h/m = r_s v_2 \beta$ or $H/m = r_s v_2$ was calculated at time $t = 0$.

Table V shows the bending angle of light numerically determined by FT with the adjustment for increasing initial photon speeds. For an initial speed equal greater than or equal to $0.9999c$, the predicted numerical value agrees to three decimal places with the celebrated analytical result.

Figure 5 shows the graph of the simulated bend angle versus time for FT with the adjustment for the case $v = 0.99999c$. From the graph, it is clear how the bend angle approaches the value of 8.534×10^{-6} rad asymptotically as the photon travels away from the sun.

Table VI shows that FT without the adjustment predicts a bend angle of precisely zero. In contrast, NT without the adjustment predicts a bend angle that is precisely 50% smaller than the bend angle that GR predicts. This result is well-known. Notably, with the space-time adjustment, FT predicts precisely the same bending of light that GR predicts in addition to precisely the same precession of Mercury that GR predicts, per Table IV. Furthermore, it is interesting that NT with no adjustment and with the adjustment,

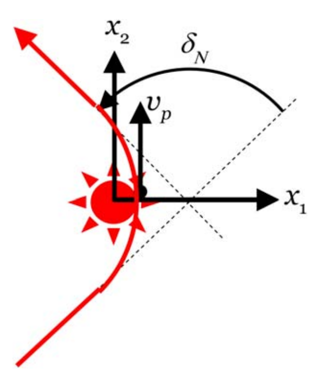

FIG. 4. (Color online) Bending of light.

f)Wikipedia, Schwarzschild geodesics, [Online]. Available: https://en.wikipedia.org/wiki/Schwarzschild_geodesics

g)Wikipedia, Gravitational lens, [Online]. Available: https://en.wikipedia.org/wiki/Gravitational_lens

TABLE V. Predicted bending of light around the sun for FT with the adjustment as a function of photon speed.

Photon velocity at perihelion	δ_N (rad)	δ_N (arc-sec)	% difference from GR
0.99c	8.620×10^{-6}	1.778	+1.008
0.999c	8.542×10^{-6}	1.762	+0.094
0.9999c	8.534×10^{-6}	1.760	0
0.99999c	8.534×10^{-6}	1.760	0

TABLE VI. Bend angle predicted by each formulation.

	Bend angle		
Formulation	rad	arc-sec	% difference from GR
a) FT without the adjustment	0	0	—
b) FT with the adjustment	8.534×10^{-6}	1.760	0
c) NT without the adjustment	4.266×10^{-6}	0.878	−50.0
d) NT with the adjustment	1.280×10^{-5}	2.640	+50.0

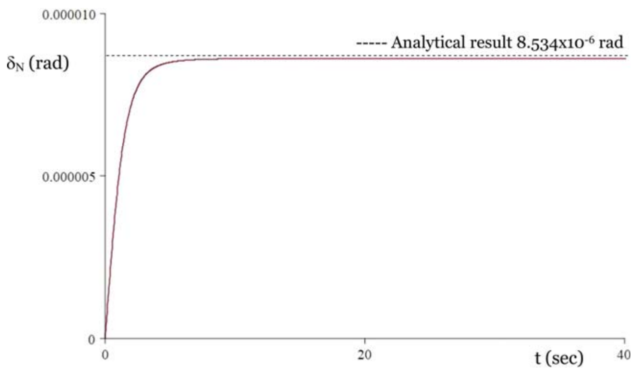

FIG. 5. (Color online) Determining the light bending angle in FT with the adjustment ($v = 0.99999c$).

respectively, predicts precisely 50% less and then precisely 50% more bending of light than GR predicts.

V. SUMMARY

This article introduced a new FT formulation that begins with a vector field of energy that satisfies vector continuity equations and that one builds up from fragments of energy. The theory is explained in detail in Appendix A. The article also introduced a space-time adjustment for the one-body gravitational problem. The article showed that the new FT with the space-time adjustment predicts the same precession of Mercury and the same bending of light that GR predicts. Independent of these results, this article also showed, without the adjustment, that FT and NT predict no precession of Mercury. In the bending of light problem without the adjustment, NT predicts a bend angle that is precisely 50% less than the bending angle of light that GR predicts and, with the adjustment, a bend angle that is precisely 50% greater than the bending angle of light that GR predicts. The equations of motion are quite simple to program, and curvilinear coordinates are not involved at all. Undergraduate students can employ the adjustment of the gravitational law to predict themselves the correct precession angle of Mercury and the correct bending angle of light. They would simply numerically integrate the orbital equations given in Table III, acquiring the physical constants from Table II.

The comparisons presented in this article highlight the intimate connection that exists between FT, NT, and GR. GR was the first theory to employ curved space-time and the first to correctly predict the precession of orbiting bodies and the bending of light around massive bodies. Both of these facts together led some to believe, quite naturally, that GR must be the only way to predict accurately the precession of Mercury and the bending of light. This article shows with a space-time adjustment that FT predicts the precession of Mercury and the bending of light precisely (and more easily than by GR). Finally, note that the authors confirmed empirically the space-time adjustment but did not rigorously derive it. The adjustment appears to be a space-time curvature term that accommodates light bending. However, without a rigorous derivation, for the time being, one must leave the space-time adjustment open to different interpretations.

Pertaining to the development of the new FT formulation, the reader may find the vector continuity requirement in FT of the vector field of energy and of the fragments of energy to be an appealing way to begin the formulation of a field theory from both axiomatic and intuitive perspectives. Axiomatically, FT parallels the tenets of analysis by beginning with a precise definition of energy over space-time (the whole) and of fragments of energy (the component parts) and then proceeds deductively from there. The theory proceeds with precise definitions of action, of interaction, from which the theory formulates change equations (which reduce to Newton's law of inertia). Intuitively, there is an important connection between the fragment of energy and the particle and wave conceptions. In NT and EM, the particle conception corresponds to energy (mass and charge). It exists at a source point and nowhere else. The wave conception corresponds to energy existing everywhere else while not explicitly addressing the source point. The particle and the wave conceptions are existential opposites, and the fragment conception is a union of the particle and the wave conceptions. Finally, in future work scientists may confirm that the new FT formulation is applicable to many more problems, including those problems that are at different scales and those problems that have complex geometries. Insightful scientists may also find that the new FT formulation unifies the representation of physical behavior. The fragment of energy is potentially a universal conception at a scale of interest of an element of energy that joins the other universal conceptions already found in physics—that of space, of time, and of energy as a whole. Only the future will tell.

APPENDIX A: NEW FT FORMULATION

Notation: Helvetica-Roman bold font for space-time vectors (e.g., **A**), Times-Roman bold font for spatial vectors (e.g., **A**), and italics for scalars (e.g., A)

Terminology (Symbols) [Universal units (in terms of length L and time T)]	
Action (P_r) [$1/L$]	Path derivative of a fragment of energy
Charge (q) [L]	Electromagnetic intensity
Conservation	Satisfies conservation law (derivable from vector continuity equations)
Continuity	Short for scalar continuity or vector continuity
Curl (λ_{rs}) [$1/L$]	Space-time matrix for electromagnetic action
Curvature (k_r) [$1/L$]	Path derivative of space-time unit vector
Electromagnetism	Discipline of macroscale science describing electrical and magnetic interactions between particles of mass and charge
Electrical field (E_r) [$1/L$]	Spatial vector function from the stationary charges of particles
Energy (A or A_r) [1]	Short for scalar field of energy or vector field of energy
Field	Spatial function or space-time function
Force (P_r or F_r)	Space-time action or space-time interaction
Fragment of energy (A_r)	Element of energy that satisfies vector continuity
Inertial reference frame	Reference frame for which fragment moves in a straight line in absence of an action upon it
Intensity (α)[L]	Corresponds to the fragment of energy
Interaction (F_r) [1]	Resultant action on a given fragment by other fragments multiplied by the negative of the intensity of the fragment
Magnetic field (B_r) [$1/L$]	Spatial vector function from the moving charges of particles
Mass (m) [T^2/L]	Mechanical intensity
Matter (α) [L]	Intensity of a fragment of energy
Mechanics	Discipline of macroscale science describing mechanical interactions between particles of mass
Particle	Discrete point with properties of mass and/or charge
Path derivative d/ds	With respect to a path increment ds
Potential	Scalar or vector function that satisfies potential equations (derivable from vector continuity equations)
Reference frame	Corresponds to measurements and a perspective, works with different coordinate systems
Scalar continuity	Property of a function changing incrementally with incremental changes in its independent variables (also called ordinary continuity)
Source point (x_{ar}) [L]	Space-time center point of a fragment or a test fragment
Space-time	One to three spatial coordinates and one temporal coordinate
Substance	Energy or matter
Unit vector (e_r) [1]	Corresponds to a source point of a fragment or a test fragment
Vector continuity	Vector function whose flow lines neither begin, nor end, nor cross
Wave	Scalar or vector function that satisfies wave equations (derivable from vector continuity equations)

This appendix develops the new field theory formulation (FT) independent of NT (Newtonian Theory), EM (Electromagnetism), SR (Special Relativity), and GR (General Relativity). The development employs four-dimensional field theory. Appendix I reviews the elements of four-dimensional vector fields.

1. Vector continuity

Begin with a 4D vector field that satisfies vector continuity in 4D space [Eqs. (I16) and (I18)]. The space-time field lines of the 4D vector field neither begin, nor end, nor cross. Represent a general 4D vector field that satisfies vector continuity as a linear combination of admissible 4D vector fields, called fragments of energy. As admissible functions, the fragments satisfy vector continuity equations but not the boundary conditions of any particular problem. As building blocks, the fragments of energy depart from the particle and the wave conceptions. The particle is a source located along a space-time line and not elsewhere, whereas the wave is missing the source and it exists everywhere else. In contrast, the fragment of energy has both a source point and it exists everywhere else.

Space-time vector continuity and the fragment of energy are generalizations of other concepts. The vector continuity equations reduce to potential equations [Eq. (I18a)], to wave equations [Eq. (I18c)], and to conservations laws [Eq. (I22)]. Furthermore, space-time vector continuity applies to systems that undergo conformal transformations in space-time.

The rectangular components of the space-time vector field and of its gradient vectors are:

$$A_1 = Ae_1, \quad A_2 = Ae_2, \quad A_3 = Ae_3, \quad A_4 = Ae_4, \tag{A1a}$$

$$\frac{\partial A_s}{\partial x_1}, \quad \frac{\partial A_s}{\partial x_2}, \quad \frac{\partial A_s}{\partial x_3}, \quad \frac{\partial A_s}{\partial x_4}, \quad (s = 1, 2, 3, 4), \tag{A1b}$$

where e_1, e_2, e_3, and e_4 are unit vector components. The differential forms of the vector continuity equations are [Eqs. (I16) and (I18)]

$$0 = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4}, \tag{A2a}$$

$$0 = \frac{\partial^2 A_s}{\partial x_1^2} + \frac{\partial^2 A_s}{\partial x_2^2} + \frac{\partial^2 A_s}{\partial x_3^2} + \frac{\partial^2 A_s}{\partial x_4^2}, \quad (s = 1, 2, 3, 4). \tag{A2b}$$

The integral forms of the vector continuity equations are [Eqs. (I16) and (I18)]

TABLE VII. Metrics.

Space	Metric	Coordinates
4D flat	$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \quad (dx_4^2 = -c^2dt^2)$ $ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) - c^2dt^2$	Rectangular Spherical
3D flat	$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$	Rectangular
4D curved[1]	$ds^2 = \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) - c^2\left(1 - \frac{r_s}{r}\right)dt^2$	Spherical (1-body)

[1]Schwarzschild metric.

$$0 = \oint (A_1 n_1 + A_2 n_2 + A_3 n_3 + A_4 n_4) dV, \tag{A3a}$$

$$0 = \oint \left(\frac{\partial A_s}{\partial x_1} n_1 + \frac{\partial A_s}{\partial x_2} n_2 + \frac{\partial A_s}{\partial x_3} n_3 + \frac{\partial A_s}{\partial x_4} n_4\right) dV, \quad (s = 1, 2, 3, 4). \tag{A3b}$$

Equations (A2a) and (A3a) correspond to Eq. (A1a), and Eqs. (A2b) and (A3b) correspond to Eq. (A1b). By counting field lines, the vector continuity equations ensure that field lines neither begin, nor end, nor cross. Equation (A3a) counts the number of field lines of $A_s \ (s = 1,\ 2,\ 3,\ 4)$ that enters a domain D as negative, and the number that leaves it as positive. In this way, the integral around the boundary of D is zero when every field line that enters D also leaves it. When Eq. (A3a) is satisfied over an arbitrary domain contained in a given region, one says that the vector field $A_s \ (s = 1, 2, 3, 4)$ satisfies vector continuity in that region.

2. Metrics

Physical theories differ by the metrics they employ. Table VII shows several of the important metrics. For FT, we shall use the 4D flat metric and, in NT, one uses the 3D flat metric. In the one-body gravitational problem, GR uses the given 4D curved metric.

3. Fragments of energy

Table VIII lists two analytical forms of 4D vector fields $A_s \ (s = 1,\ 2,\ 3,\ 4)$ that satisfy space-time vector continuity [Eqs. (A2) and (A3)]. As shown, the fragment of energy assumes the form of the reciprocal of spatial distance. One can show that this is *not* coincidental (see Appendix D).

4. General form of the solution

Any linear combination of admissible vector fields is an admissible vector field

$$A_s = \alpha_a A_{as} + \alpha_b A_{bs} + \alpha_c A_{cs} + \cdots. \tag{A4}$$

Refer to the coefficients $\alpha_a,\ \alpha_b$, etc., in the linear combination as the intensities of the fragments. The vector field $A_s \ (= 1, 2, 3, 4)$ satisfies vector continuity. However, as stated earlier, it does not naturally satisfy boundary conditions. A uniqueness theorem can determine the boundary conditions that produce a unique vector field solution $A_s \ (s = 1, 2, 3, 4)$ (see Appendix C).

5. Action and interaction

Consider the vector field $A_s \ (s = 1, 2, 3, 4)$ of a single fragment of energy. Its source point is located at $x_{0s} \ (s = 1, 2, 3, 4)$, and the path of its source point is along the unit vector components $e_s = dx_{0s}/ds \ (s = 1, 2, 3, 4)$ [Eq. (I4)]. If the fragment is initially at rest, it will remain at rest in the absence of being acted on by another fragment *only if* the change is observed in an inertial frame of reference. Only in the inertial frame does the change in the space-time direction of the fragment *not* result from itself.

Action: The action by a fragment is the path derivative

$$P_s = \frac{dA_s}{ds}. \tag{A5}$$

Action is a 4D vector field that mathematically decomposes into a mechanical part and an electromagnetic part

$$P_s = S_s + T_s, \tag{A6}$$

TABLE VIII Fragment of energy and wave.

	Fragment of energy and wave (admissible vector fields)				
Type	A	e_1	e_2	e_3	e_4
Fragment of energy[1]	$\frac{1}{r}$	0	0	0	1
Wave[1]	$A(u)$	$e_1(u)$	$e_2(u)$	$e_3(u)$	$e_4(u)$

[1]In fundamental units

$$r^2 = (x_1 - x_{a1})^2 + (x_2 - x_{a2})^2 + (x_3 - x_{a3})^2$$
$$u = u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4, \quad u_1^2 + u_2^2 + u_3^2 + u_4^2 = 0, \quad u_4 = \pm i$$
$$x_1 = u_1ct, \quad x_2 = u_2ct, \quad x_3 = u_3ct$$

$u_r \ (r = 1,\ 2,\ 3,\ 4)$ are constants. $x_{a1},\ x_{a2},$ and x_{a3} are the components of source point of the fragment of energy.

$$S_s = \frac{\partial A}{\partial x_s} \ , \tag{A7}$$

$$T_s = -\left(\frac{\partial A_1}{\partial x_s} - \frac{\partial A_s}{\partial x_1}\right)e_1 - \left(\frac{\partial A_2}{\partial x_s} - \frac{\partial A_s}{\partial x_2}\right)e_2 - \left(\frac{\partial A_3}{\partial x_s} - \frac{\partial A_s}{\partial x_3}\right)e_3 - \left(\frac{\partial A_4}{\partial x_s} - \frac{\partial A_s}{\partial x_4}\right)e_4. \tag{A8}$$

Equation (A7) is the mechanical part (notice its conservative form), and Eq. (A8) is the electromagnetic part (notice its Lorentzian form). For the mathematical derivation of Eqs. (A6)–(A8), see Appendix E.

Interaction: Action is a vector field that is *by* a fragment, also called an action force, and interaction is a vector that acts *on* a fragment, also called an interaction force. Consider fragments a, b, c, d, etc. Define the interaction force acting on fragment a by fragments b, c, d, etc., to be the resultant of the action forces by fragments b, c, d, etc., evaluated at the source point of fragment a multiplied by the negative of the intensity α_a of fragment a. Thus, the resultant of the action forces by fragments b, c, d, etc., evaluated at the center of fragment a is

$$P_{as}|_a = (P_{bs} + P_{cs} + P_{ds} + \cdots)|_a. \tag{A9}$$

The interaction force acting on fragment a by fragments b, c, d, etc., is

$$F_{as} = -\alpha_a P_{as}|_a, \ \ \alpha_a = \begin{cases} m_a c^2 & \textit{mechanical}, \\ q_a & \textit{electromagnetic}. \end{cases} \tag{A10}$$

One sees in Eq. (A10) that $P_{as}|_a$ ($s = 1, 2, 3, 4$) is a resultant of other fragments. Also, note that one can drop the evaluation sign without causing confusion because P_{as} $(s = 1, 2, 3, 4)$ in Eq. (A10) is never evaluated at its own source point. Furthermore, notice that one associates the fragment's intensity α_a as with either mechanical behavior or as with electromagnetic behavior. The association depends on the nature of the change in Eq. (A6).

Finally, by their forms (see Table VIII), it follows that mechanical interaction forces between any pair of fragments are central forces. They act in equal and opposite directions along the line between their source points. Designating two fragments by a and b, one expresses these properties as

$$\begin{aligned} &F_{as} + F_{bs} = 0 \\ &F_{as} = C(x_{as} - x_{bs}) \ \text{for some} \ C \neq 0 \end{aligned} \quad (s = 1, 2, 3, 4). \tag{A11}$$

6. The change equation

An essential part of analysis is to understand how to divide a whole into disconnected parts and then to reconnect the parts to reform the whole. The concepts of action force and interaction force are tools that facilitate analysis. In an inertial space, by assuming that a fragment that is initially at rest will remain at rest, we had divided a whole into its disconnected parts. Each fragment of energy, when disconnected from the other, was following a straight space-time line as though it belonged to a separate, unconnected world. In the disconnected state, the action forces on the fragments were set to zero. The change equations reconnect the fragments to each other to reform the whole. One expresses the change in the straight-line path of the source point of a fragment by curvature vector components $k_s = \frac{de_s}{ds}$ $(s = 1, 2, 3, 4)$ [Eq. (I8)]. The change equations are

$$\frac{dA_{bs}}{ds_b} + \frac{dA_{cs}}{ds_c} + \frac{dA_{ds}}{ds_d} + \cdots = k_{as} \ \ (s = 1, 2, 3, 4). \tag{A12}$$

Equation (A12) sets the resultant action force acting on a fragment to the curvature vector of the fragment's source point. Under the nonrelativistic assumption, the change equation reduces to Newton's law of inertia (see Appendix F). Newton's law of inertia is

$$F_{bs} + F_{cs} + F_{ds} + \cdots = \ m_a a_{as} \quad (s = 1, 2, 3). \tag{A13}$$

The following describes momentum and energy in the FT formulation, each of which reduce to their classical counterparts under the nonrelativistic speed assumption [Eq. (I8c)].

7. Momentum and energy

Based on the preceding development, we now examine momentum and energy in the two-body gravitational problem (see Appendix B).

Linear momentum: From Eqs. (A5)–(A7), (A10), and (A12)

$$\begin{aligned} -\frac{F_r}{c^2} &= mk_{Cr} = m_a k_{ar} + m_b k_{br}, \\ \frac{L_r}{ic} &= me_r = m_a e_{ar} + m_b e_{br}, \\ mx_{Cr} &= m_a x_{ar} + m_b x_{br}, \end{aligned} \tag{A14a-c}$$

where $m = m_a + m_b$. Equation (A14a) defines resultant force components F_r external to the system, a resultant mass m internal to the system, and curvature components k_{Cr} of the mass center of the system. Equation (A14b) defines linear momentum components L_r of the system, and Eq. (A14c) defines coordinates x_{Cr} of the system's mass center. Under the nonrelativistic speed assumption, Eqs. (A14a)–(A14c) reduce to their classical counterparts.

Angular momentum: The cross product is a 3D operation. The more general form utilizes permutation symbols [Eq. (I5c) and Fig. 6], in terms of which a resultant moment about the mass center, a resultant angular momentum about the mass center, and the relationship between them are

$$M_{Cr} = \varepsilon_{rst4}(x_{as}F_{at} + x_{bs}F_{bt}), \tag{A15a}$$

$$\frac{h_{Cr}}{ic} = \varepsilon_{rst4}(x_{as}m_a e_{at} + x_{bs}m_b e_{bt}), \quad ic\frac{dh_{Cr}}{ds} = M_{Cr} = 0. \tag{A15b,c}$$

Equation (A15a) defines resultant moment components M_{Cr}, and Eq. (A15b) defines angular momentum

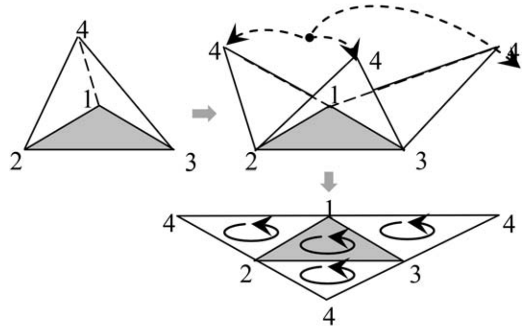

FIG. 6. Right-hand rule for 4D space.

components h_{Cr}. Under the nonrelativistic assumption, Eqs. (A15) reduce to their classical counterparts.

Energy: The kinetic energy T is

$$T = \frac{1}{2}mv^2\frac{1}{1-\left(\frac{v}{c}\right)^2}\quad\left(\cong\frac{1}{2}mv^2\right). \tag{A16}$$

Conservation of energy is determined as follows:

$$\begin{aligned}0 &= -\frac{mc^2}{2}\int\frac{d}{ds}\left(e_1^2+e_2^2+e_3^2+e_4^2\right)ds\\ &= -mc^2\int(k_1e_1+k_2e_2+k_3e_3+k_4e_4)ds\\ &= \int(F_1e_1+F_2e_2+F_3e_3+F_4e_4)ds\\ &= \int(F_1e_1+F_2e_2+F_3e_3)ds-\mathrm{m}c^2\int k_4e_4ds\\ &= \int\left(-\frac{\partial V}{\partial x_1}dx_1-\frac{\partial V}{\partial x_2}dx_2-\frac{\partial V}{\partial x_3}dx_3\right)-\frac{mc^2}{2}\int d\left(e_4^2\right),\\ &= V_1-V_2-\frac{mc^2}{2}\left(\frac{1}{1-\left(\frac{v_2}{c}\right)^2}-\frac{1}{1-\left(\frac{v_1}{c}\right)^2}\right)\\ &= V_1-V_2-\frac{mc^2}{2}\left(\frac{\left(\frac{v_2}{c}\right)^2}{1-\left(\frac{v_2}{c}\right)^2}-\frac{\left(\frac{v_1}{c}\right)^2}{1-\left(\frac{v_1}{c}\right)^2}\right),\end{aligned}$$

where V is potential energy. It follows that:

$$E = T+V,\quad E_1 = E_2. \tag{A17}$$

Under the nonrelativistic assumption, Eqs. (A16) and (A17) reduce to their classical counterpart.

APPENDIX B: GRAVITATIONAL EQUATIONS FROM FT

1. From the two-body problem to the one-body problem

Locate two bodies in an inertial frame. The system's mass center is located at the frame's origin, and it is stationary [$x_{Cr}=0,\ e_{Cr}=0$, and $x_{Cr}=0$ in Eq. (A14)]. It follows that the moment about the mass center is zero, too ($M_C=0$). The gravitational action forces are mechanical, that is, we assume that their electromagnetic parts cancelled [see Eqs. (A5)–(A8)]. From Eq. (A12)

$$P_{br} = \frac{\partial A}{\partial x_r} = k_{br}, \tag{B1}$$

where A is the magnitude of the energy vector field of body a. From Eq. (A15), the angular momentum of the system is constant

$$\begin{aligned}\frac{h_C}{ic} &= m_a(x_{a1}e_{a2}-x_{a2}e_{a1})+m_b(x_{b1}e_{b2}-x_{b2}e_{b1})\\ &= \text{constant}.\end{aligned} \tag{B2}$$

Next, convert this two-body problem to a one-body problem. Let

$$\begin{aligned}&\begin{matrix}m_ae_{ar}=-\mu e_r\\ m_ak_{ar}=-\mu k_r\end{matrix},\quad \begin{matrix}m_be_{br}=\mu e_r, & e_r=e_{br}-e_{ar}\\ m_bk_{br}=\mu k_r, & k_r=k_{br}-k_{ar}\end{matrix},\\ &\mu=\frac{m_am_b}{m_a+m_b}\quad(r=1,2).\end{aligned} \tag{B3}$$

Equations (B3) satisfy Eqs. (A14). From Eqs. (A10) and (A12),

$$F_{br} = -m_bc^2k_{br} = -\mu c^2k_r. \tag{B4}$$

Substitute Eqs. (B3) into Eq. (B2) to get [Eq. (I8c)]

$$\begin{aligned}&\frac{h_C}{ic}=\frac{m_am_b}{m_a+m_b}(x_1e_2-x_2e_1)=\mu\left(x_1\frac{v_2}{ic}-x_2\frac{v_1}{ic}\right),\\ &\mu=\frac{m_am_b}{m_a+m_b},\quad \beta=\left(1-\left(\frac{v}{c}\right)^2\right)^{-1/2}.\end{aligned} \tag{B5}$$

From Eq. (B5), the angular momentum is

$$h_C = \mu(x_1v_2-x_2v_1)\beta. \tag{B6}$$

In the gravitational potential below, $m_a=M$ denotes the more massive body and $m_b=m$ denotes the less massive body. We will replace h_C above with the term

$$h = m(x_1v_2-x_2v_1)\beta\quad\text{where}\quad h_C=\frac{\mu}{m}h. \tag{B7}$$

Note that an analogous substitution arises in NT, where we replace $H_C=\ \mu(x_1v_2-x_2v_1)$ with

$$H=\ m(x_1v_2-x_2v_1)\quad\text{where}\quad H_C=\frac{\mu}{m}H.$$

2. Relationship between curvature and acceleration in FT

From Eq. (I8b), one can express the three-dimensional spatial curvature vector $\mathbf{k}_3$ in terms of the three-dimensional acceleration vector by

$$\mathbf{k}_3 = \frac{1}{v^2 - c^2}\left\{\mathbf{a} - va\frac{1}{v^2 - c^2}\mathbf{v}\right\} = \frac{-\mathbf{a}}{c^2\left(1 - \left(\frac{v}{c}\right)^2\right)} - \frac{\mathbf{v}}{c^2}\left[\frac{v_1a_1 + v_2a_2 + v_3a_3}{\left(1 - \left(\frac{v}{c}\right)^2\right)^2}\right],$$

$$= \frac{1}{c^2}\frac{-1}{\left(1 - \left(\frac{v}{c}\right)^2\right)^2}\left\{\mathbf{a}\left(1 - \left(\frac{v}{c}\right)^2\right) + \mathbf{v}[v_1a_1 + v_2a_2 + v_3a_3]\right\}. \tag{B8}$$

For the planar orbital motion of Mercury about the sun,

$$\mathbf{k}_2 = \frac{-1}{c^2\left(1 - \left(\frac{v}{c}\right)^2\right)^2}\left\{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\left(1 - \left(\frac{v}{c}\right)^2\right) + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}[v_1a_1 + v_2a_2 + v_3a_3]\right\}$$

$$= \frac{-1}{c^2\left(1 - \left(\frac{v}{c}\right)^2\right)^2}\begin{bmatrix} 1 - \left(\frac{v_2}{c}\right)^2 & \left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) \\ \left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) & 1 - \left(\frac{v_1}{c}\right)^2 \end{bmatrix}\mathbf{a}, \tag{B9}$$

where

$$\begin{bmatrix} 1 - \left(\frac{v_2}{c}\right)^2 & \left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) \\ \left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) & 1 - \left(\frac{v_1}{c}\right)^2 \end{bmatrix}^{-1} = \frac{1}{D}\begin{bmatrix} 1 - \left(\frac{v_1}{c}\right)^2 & -\left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) \\ -\left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) & 1 - \left(\frac{v_2}{c}\right)^2 \end{bmatrix}.$$

$$D = \left(1 - \left(\frac{v_1}{c}\right)^2\right)\left(1 - \left(\frac{v_2}{c}\right)^2\right) - \left(\frac{v_1}{c}\frac{v_2}{c}\right)^2 = 1 - \left(\left(\frac{v_1}{c}\right)^2 + \left(\frac{v_2}{c}\right)^2\right) = 1 - \left(\frac{v}{c}\right)^2$$

It follows that:

$$\mathbf{a} = -c^2\left(1 - \left(\frac{v}{c}\right)^2\right)^2\frac{1}{1 - \left(\frac{v}{c}\right)^2}\begin{bmatrix} 1 - \left(\frac{v_1}{c}\right)^2 & -\left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) \\ -\left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) & 1 - \left(\frac{v_2}{c}\right)^2 \end{bmatrix}\mathbf{k}_2,$$

$$= -c^2\left(1 - \left(\frac{v}{c}\right)^2\right)\begin{bmatrix} 1 - \left(\frac{v_1}{c}\right)^2 & -\left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) \\ -\left(\frac{v_1}{c}\right)\left(\frac{v_2}{c}\right) & 1 - \left(\frac{v_2}{c}\right)^2 \end{bmatrix}\mathbf{k}_2. \tag{B10}$$

Substituting the action force in Table I into Eq. (B4) and substituting the result into Eq. (B10) yield the acceleration in Table III for the new FT formulation.

APPENDIX C: THE UNIQUENESS THEOREM FOR 4D VECTOR FIELDS

Distinguish between two solutions of the vector continuity equations (A2b) by a superscript and express the difference between the two solutions as

$$A_s = A_s^{(1)} - A_s^{(2)}, \quad (s = 1,\ 2,\ 3,\ 4). \tag{C1}$$

Next, consider the mathematical identity [Eqs. (I12) and (I13)]

$$0 = \int \left(\frac{\partial C_1}{\partial x_1} + \frac{\partial C_2}{\partial x_2} + \frac{\partial C_3}{\partial x_3} + \frac{\partial C_4}{\partial x_4} \right) dx_1 dx_2 dx_3 dx_4$$
$$= \oint (C_1 n_1 + C_2 n_2 + C_3 n_3 + C_4 n_4) dx_1 dx_2 dx_3, \quad \text{(C2)}$$

where

$$C_s = A_1 \frac{\partial A_1}{\partial x_s} + A_2 \frac{\partial A_2}{\partial x_s} + A_3 \frac{\partial A_3}{\partial x_s} + A_4 \frac{\partial A_4}{\partial x_s}, \quad (s = 1, 2, 3, 4)$$

It follows that:

$$\frac{\partial C_1}{\partial x_1} + \frac{\partial C_2}{\partial x_2} + \frac{\partial C_3}{\partial x_3} + \frac{\partial C_4}{\partial x_4} = \sum_{s=1}^{s=4} \sum_{t=1}^{t=4} \left(\frac{\partial A_s}{\partial x_t} \right)^2, \quad \text{(C3)}$$

$$C_1 n_1 + C_2 n_2 + C_3 n_3 + C_4 n_4 = A_1 \frac{dA_1}{dn} + A_2 \frac{dA_2}{dn} + A_3 \frac{dA_3}{dn} + A_4 \frac{dA_4}{dn}. \quad \text{(C4)}$$

Substituting Eqs. (C3) and (C4) into Eq. (C2)

$$0 = \int \left(\sum_{s=1}^{s=4} \sum_{t=1}^{t=4} \left(\frac{\partial A_s}{\partial x_t} \right)^2 \right) dx_1 dx_2 dx_3 dx_4,$$
$$= \oint \left(A_1 \frac{dA_1}{dn} + A_2 \frac{dA_2}{dn} + A_3 \frac{dA_3}{dn} + A_4 \frac{dA_4}{dn} \right) dx_1 dx_2 dx_3. \quad \text{(C5)}$$

First, consider in Eq. (C5) the integrand in the event integral. Notice that it is equal to zero only if all of the partial derivatives vanish implying, for uniqueness, that A_s must be constant throughout the region. It follows, for uniqueness, that the two solutions $A_s^{(1)}$ and $A_s^{(2)}$ can only be unique up to an additive constant. Next, consider the integrand in the boundary integral. It dictates the quantities that one must specify in order to obtain a solution that is unique up to an additive constant. To obtain the unique solution, one must specify either A_s or dA_s/dn for each $s = 1, 2, 3, 4$ at each point on the boundary. It is interesting to imagine the region to be the entire universe, that it contains no singularities, and that A_s or dA_s/dn is equal to zero at its boundary (past and present). That would imply that the universe is void of change. Fortunately, the universe contains singularities—the source points of the fragments.

APPENDIX D: THE UNIQUE PROPERTY OF THE FUNCTION *A* = 1/*r*

Begin with the differential form of the vector continuity equations of $A_s = Ae_s \ (s = 1, 2, 3, 4)$ and of its gradient vectors, expressed in rectangular coordinates as [Eqs. (A2) and (A3)]

$$0 = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4}, \quad \text{(D1)}$$

$$0 = \frac{\partial^2 A_s}{\partial x_1^2} + \frac{\partial^2 A_s}{\partial x_2^2} + \frac{\partial^2 A_s}{\partial x_3^2} + \frac{\partial^2 A_s}{\partial x_4^2}, \quad (s = 1, 2, 3, 4). \quad \text{(D2)}$$

Consider a stationary fragment for which $e_s = \delta_{s4}$ $(s = 1, 2, 3, 4)$. (δ_{rs} is the Kronecker-delta function for which $\delta_{rs} = 1$ when $r = s$ and $\delta_{rs} = 0$ otherwise) [Eq. (I23)]. Thus, for the stationary fragment, $A_1 = A_2 = A_3 = 0$ and $A_4 = A$. From Eq. (D1), $\partial A/\partial x_4 = 0$ so A is a not a function of time, only of the spatial coordinates. Next, for the purposes of the fragment serving as a building block, disallow giving preference to any one spatial direction, that is, disallow directionality. The vector field, under this assumption, becomes a function of $r = \left[(x_1 - x_{01})^2 + (x_2 - x_{02})^2 + (x_1 - x_{03})^2 \right]^{1/2}$, where x_{0s} $(s = 1,\ 2,\ 3)$ are the coordinates of the fragment's point source. Perform the following mathematical steps:

$$\frac{\partial A}{\partial x_s} = \frac{dA}{dr} \frac{\partial r}{\partial x_s} = \frac{x_s - x_{0s}}{r} \frac{dA}{dr} \quad (s = 1, 2, 3), \quad \frac{\partial A}{\partial x_4} = 0,$$
$$\frac{\partial^2 A}{\partial x_s^2} = \frac{\partial}{\partial x_s} \left(\frac{x_s - x_{0s}}{r} \frac{dA}{dr} \right) = \left(\frac{x_s - x_{0s}}{r} \right)^2 \frac{d^2 A}{dr^2} + \frac{\left(r^2 - (x_s - x_{0s})^2 \right)}{r^3} \frac{dA}{dr} (s = 1, 2, 3), \quad \frac{\partial^2 A}{\partial^2 x_4} = 0.$$

From Eq. (D2),

$$0 = \left[\left(\frac{x_1 - x_{01}}{r} \right)^2 \frac{d^2 A}{dr^2} + \frac{\left(r^2 - (x_1 - x_{01})^2 \right)}{r^3} \frac{dA}{dr} \right] + \left[\left(\frac{x_2 - x_{02}}{r} \right)^2 \frac{d^2 A}{dr^2} + \frac{\left(r^2 - (x_2 - x_{02})^2 \right)}{r^3} \frac{dA}{dr} \right] + \left[\left(\frac{x_3 - x_{03}}{r} \right)^2 \frac{d^2 A}{dr^2} + \frac{\left(r^2 - (x_3 - x_{03})^2 \right)}{r^3} \frac{dA}{dr} \right]$$
$$= \frac{d^2 A}{dr^2} + \frac{2}{r} \frac{dA}{dr},$$

so

$$0 = \frac{dB}{dr} + \frac{2}{r} B, \quad \text{(D3)}$$

in which $B = dA/dr$. Finally, solve Eq. (D3) to get $B = \alpha/r^2$. The form of the stationary fragment becomes

$$A = -\frac{\alpha}{r} + C, \quad \text{(D4)}$$

where α and C are constants and where one refers to α as the fragment's intensity.

TABLE IX. Mechanical constants and electromagnetic constants.

Mechanical constants	Electromagnetic constants
Universal gravitational constant: $G = 6.67 \times 10^{-11}\ \mathrm{m^3/kg \cdot s^2}$	Speed of light: $c = 3 \times 10^9\ \mathrm{m/s}$
	Electric constant: $k = 8.99 \times 10^9\ \mathrm{m^3kg/C^2s^2}$
Electron mass: $m_e = 9.1 \times 10^{-31}\ \mathrm{kg}$	Electron charge: $q_e = 1.6 \times 10^{-19}\ C$

APPENDIX E: DECOMPOSITION OF ACTION INTO A MECHANICAL PART AND AN ELECTROMAGNETIC PART

Consider the 4D vector field components $A_s = Ae_s$ ($s = 1, 2, 3, 4$). The components of its path derivative in the direction of its unit vector are $P_s = dA_s/ds$, where $dx_s = e_s ds$ ($s = 1, 2, 3, 4$). The following shows that one can express the components of the path derivative as:

$$P_s = S_s + T_s, \quad (\mathrm{s} = 1, 2, 3, 4), \tag{E1}$$

in which

$$S_s = \frac{\partial A}{\partial x_s}, \tag{E2}$$

$$T_s = -\ \lambda_{s1}e_1 - \lambda_{s2}e_2 - \lambda_{s3}e_3 - \lambda_{s4}e_4. \tag{E3}$$

We shall refer to the path derivative as action or as action force. In Eq. (E1), $S_s(\mathrm{s} = 1, 2, 3, 4)$ are the components of the mechanical part of the action force, $T_s(\mathrm{s} = 1, 2, 3, 4)$ are the components of the electromagnetic part of the action force, and $\lambda_{st} = (\partial A_t/\partial x_s) - (\partial A_s/\partial x_t)$, ($s, t = 1, 2, 3, 4$) are the entries of a curl matrix [Eq. (G5)].

To show Eq. (E1), adopt the summation notation for repeated indices and manipulate P_s ($s = 1, 2, 3, 4$) as follows:

$$\begin{aligned}
P_s &= \frac{dA_s}{ds} = \frac{\partial A_s}{\partial x_t}\frac{dx_t}{ds} = \frac{\partial A_t}{\partial x_s}e_t - \left(\frac{\partial A_t}{\partial x_s} - \frac{\partial A_s}{\partial x_t}\right)e_t = S_s + T_s,\\
S_s &= \frac{\partial A_t}{\partial x_s}e_t = \frac{\partial (Ae_t)}{\partial x_s}e_t = A\frac{\partial e_t}{\partial x_s}e_t + \frac{\partial A}{\partial x_s}e_te_t = \frac{\partial A}{\partial x_s},\\
T_s &= -\left(\frac{\partial A_t}{\partial x_s} - \frac{\partial A_s}{\partial x_t}\right)e_t = -\ \lambda_{st}e_t,
\end{aligned} \tag{E4}$$

since $(\partial e_t/\partial x_s)e_t = \partial/\partial x_s(e_te_t) = 0$ and $e_te_t = 1$. This completes the derivation of Eq. (E1). However, let us also put the electromagnetic part of the action into its more familiar classical Lorentzian vector form $\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$, where quantities in bold are 3D vectors. Write the electromagnetic interaction force as $F_s = -\alpha\ T_s = q\ \lambda_{st}e_t$. This yields the classical Lorentzian vector form

$$\begin{aligned}
\mathbf{F} &= q\begin{bmatrix} -i\mathbf{B}\times & \mathbf{E} \\ -\mathbf{E}\cdot & 0 \end{bmatrix}\begin{pmatrix} \mathbf{e} \\ e_4 \end{pmatrix} = q\begin{bmatrix} -i\mathbf{B}\times & \mathbf{E} \\ -\mathbf{E}\cdot & 0 \end{bmatrix}\begin{pmatrix} \mathbf{v}/ic \\ 1 \end{pmatrix}\\
&= q\begin{pmatrix} -i\mathbf{B} \times \dfrac{\mathbf{v}}{ic} + \boldsymbol{E} \\ -\mathbf{E}\cdot\mathbf{v}/ic \end{pmatrix} = q\begin{pmatrix} \mathbf{E} + \dfrac{1}{c}\mathbf{v} \times \mathbf{B} \\ -\mathbf{E}\cdot\mathbf{v}/ic \end{pmatrix},\\
\mathbf{B} &= \begin{bmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{bmatrix}, \quad \mathbf{E}\cdot = \begin{pmatrix} E_1 & E_2 & E_3 \end{pmatrix}.
\end{aligned} \tag{E5}$$

APPENDIX F: FROM THE CHANGE EQUATION TO NEWTON'S LAW OF INERTIA

Below we deduce the law of inertia $F_s = ma_s$, ($s = 1, 2, 3$) [Eq. (A13)] from the change equation $P_s = k_s$, ($s = 1, 2, 3, 4$) [Eq. (A12)]. First, invoke the relationship $k_s = -a_s/c^2$, ($s = 1, 2, 3$) under the nonrelativistic speed assumption [Eq. (I8c)], where a_s, ($s = 1, 2, 3$) are acceleration components. Next, designate the fragment's intensity as either mechanical or electromagnetic

$$\alpha = \begin{cases} mc^2 & \text{mechanical} \\ q & \text{electromagnetic} \end{cases}, \tag{F1}$$

where m is mass and q is charge. Substitute these into Eq. (A10), which is the expression for the interaction force, to get

$$F_s = \ ma_s \text{ mechanical}, \tag{F2}$$

$$F_s = \left(\frac{q}{c^2}\right)a_s \text{ electromagnetic}. \tag{F3}$$

Equation (F2) is the "mechanical" law of inertia, and Eq. (F3) is the "electromagnetic" law of inertia. In NT and EM, one always adopts the mechanical law of inertia in both mechanics and in electromagnetism. Let us explain the reason for adopting the mechanical law in electromagnetism.

One explains this based on how, at the atomic scale, elements bundle. Consider a pair of electrons. Begin by converting quantities expressed in terms of conventional units into universal units. Table IX lists the physical constants, and Table X expresses three classical laws in terms of conventional units and universal units.

In Table X, the subscript 0 indicates an expression in terms of universal units. One expresses acceleration a and distance r in terms of length L and time T, whether in universal units or in conventional units, so their expressions are the same in both conventional units and universal units. One finds from Table X: $F/F_0 = m/m_0$, $F/F_0 = F_e/F_{e0} = F_g/F_{g0}$, and $m/m_0 = m_e/m_{e0}$ so the conversions are

$$\frac{q_e}{q_{e0}} = \frac{c^2}{\sqrt{Gk}}, \quad \frac{m_e}{m_{e0}} = \frac{c^4}{G}\ . \tag{F4}$$

From Table IX, the ratio of mechanical intensity to electromagnetic intensity and the ratio of gravitational interaction to electromagnetic interaction are

$$\frac{\alpha_m}{\alpha_e} = \frac{m_ec^2}{q_e} = 5.12 \times 10^7, \tag{F5}$$

TABLE X. Fundamental unit and conventional unit comparison.

Equation	Universal units	Conventional units
Newton's second law	$F_0 = m_0 a$	$F = ma$
Universal law of gravitation	$F_{g0} = c^4 \frac{m_{e0}^2}{r^2}$	$F_g = G\frac{m_e^2}{r^2}$
Coulomb law of electrostatics	$F_{e0} = \frac{q_{e0}^2}{r^2}$	$F_e = k\frac{q_e^2}{r^2}$

$$\frac{F_g}{F_e} = \frac{G}{k}\left(\frac{m_e}{q_e}\right)^2 = 2.34 \times 10^{-43}. \tag{F6}$$

Equations (F5) and (F6) show that, for a pair of electrons, mass dominates over charge while electromagnetic interaction force dominates over gravitational interaction force. One expresses these relationships as the two properties

$$\begin{cases} P_s = S_{ms} + T_{es} \cong T_{es} \quad (S_{es} \cong 0), \\ \alpha = \alpha_m + \alpha_e \cong \alpha_m \qquad (\alpha_q \cong 0). \end{cases} \tag{F7}$$

Per Eq. (F7), in the electron's bundle, its mass dominates over its charge, and its electromagnetic action force dominates over its gravitational action force, both for reasons that are associated with the nature of the bundle at the atomic scale, which is beyond the scope of this article. The electron's aggregate curvature is $k_s = 1/\alpha(\alpha_m k_{ms} + \alpha_q k_{qs}) = (\alpha_e/\alpha_m)T_s = (q/mc^2)T_s$. Finally, under the nonrelativistic speed assumption, $m\mathbf{a} = -q\mathbf{T} = q(\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{B})$, which is consistent with the experimentally verified motion of a charged particle. Thus, one employs Eq. (F2) and not Eq. (F3) in both NT and EM (at the usual scales).

APPENDIX G: FROM VECTOR CONTINUITY TO MAXWELL'S EQUATIONS

Consider a domain that contains a fragment of energy A_{0s} ($s = 1, 2, 3, 4$) that excites a vector field of energy with components A_s ($s = 1, 2, 3, 4$) in a given domain. Other so-called excitations could exist outside of the domain. One would treat their influences on the components A_s ($s = 1, 2, 3, 4$) of the vector field of energy in the domain through boundary conditions. Next, assume that one is observing continuity in an inertial frame of reference. All of the considerations are in a single frame of reference. One has no need to invoke RT. Finally, assume that A_s ($s = 1, 2, 3, 4$) and the components of its gradient vectors satisfy space-time continuity except at the source point of the fragment of energy A_{0s} ($s = 1, 2, 3, 4$) [Eqs. (I12), (I13), and (I29)]

$$\begin{array}{lll} \text{vector continuity} & \text{integral form} & \text{differential form} \\ A_s & 0 = \oint \mathbf{A}\cdot\mathbf{n}dV & \Box\cdot\mathbf{A} = 0 \\ \Box A_s \ (s = 1,2,3,4) & 4\pi\alpha_s \int \delta_3 dE = \oint \Box A_s\cdot\mathbf{n}dV & 4\pi\alpha_s\delta_3 = \Box\cdot\Box A_s \end{array}. \tag{G1a,b}$$

In Eq. (G1), we switched to a vector notation consistent with the customary notation of Maxwell's equations. To derive Maxwell's equations, particular interest will lie in vector continuity of the curl vectors $\Lambda_s = (\partial A_s/\partial\mathbf{x}) - (\partial\mathbf{A}/\partial x_s)$, ($s = 1, 2, 3, 4$). From Eq. (G1), the space-time vector continuity equations of the curl vectors are

$$\begin{array}{ll} \text{integral form} & 4\pi\alpha_s \int \delta_3 dE = \oint \Lambda_s\cdot\mathbf{n}dV, \\ \text{differential form} & 4\pi\alpha_s\delta_3 = \sum_{t=1}^{t=4}\frac{\partial\lambda_{ts}}{\partial x_t}. \end{array} \tag{G2a,b}$$

In addition to vector continuity of the curl vectors, we will call upon the curl identities. Their integral and differential forms are [Eqs. (I10) and (I11)]

$$\begin{array}{ll} \text{integral form} & 0 = \oint_{(r,s,t)} \lambda_{rs}dx_r dx_s \\ \text{differential form} & 0 = \omega_{rst} = \frac{\partial\lambda_{rs}}{\partial x_t} + \frac{\partial\lambda_{st}}{\partial x_r} + \frac{\partial\lambda_{tr}}{\partial x_s} \end{array}$$
$$(r,s,t) = \begin{cases} (2,4,3) & \text{for} \quad u = 1 \\ (3,4,1) & \text{for} \quad u = 2 \\ (4,2,1) & \text{for} \quad u = 3 \\ (1,2,3) & \text{for} \quad u = 4 \end{cases}. \tag{G3a,b}$$

Show that Eqs. (G2) and (G3) generate Maxwell's equations

$$\begin{array}{ll} \text{integral form} & \text{differential form} \\ \frac{4\pi}{c}\int \mathbf{j}\cdot\mathbf{n}dS = \oint \mathbf{B}\cdot d\mathbf{l} - \frac{1}{c}\frac{\partial}{\partial t}\int \mathbf{E}\cdot\mathbf{n}dS & \frac{4\pi}{c}\mathbf{j}\delta = \nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \\ 4\pi q = \oint \mathbf{E}\cdot\mathbf{n}dS & 4\pi q\delta = \nabla\cdot\mathbf{E} \\ 0 = \oint \mathbf{E}\cdot d\mathbf{l} + \frac{1}{c}\frac{\partial}{\partial t}\int \mathbf{B}\cdot\mathbf{n}dS & 0 = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla\times\mathbf{E} \\ 0 = \oint \mathbf{B}\cdot\mathbf{n}dS & 0 = \nabla\cdot\mathbf{B} \end{array}. \tag{G4a-d}$$

Begin to show Eqs. (G4) by defining the electric field vector **E** and the magnetic field vector **B** in terms of the elements of the curl matrix $\mathbf{\Lambda}$ and rewrite the needed relationships in the convenient forms

$$\Lambda = \begin{bmatrix} 0 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & 0 & \lambda_{23} & \lambda_{24} \\ \lambda_{31} & \lambda_{32} & 0 & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 0 \end{bmatrix} = \begin{bmatrix} 0 & iB_3 & -iB_2 & E_1 \\ -iB_3 & 0 & iB_1 & E_2 \\ iB_2 & -iB_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{bmatrix},$$

$$\mathbf{\Lambda}_1 = \begin{pmatrix} 0 \\ -iB_3 \\ iB_2 \\ -E_1 \end{pmatrix} \quad \Lambda_2 = \begin{pmatrix} iB_3 \\ 0 \\ -iB_1 \\ -E_2 \end{pmatrix} \quad \Lambda_3 = \begin{pmatrix} -iB_2 \\ iB_1 \\ 0 \\ -E_3 \end{pmatrix}$$

$$\mathbf{\Lambda}_4 = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ 0 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} \lambda_{14} \\ \lambda_{24} \\ \lambda_{34} \end{pmatrix} \quad i\mathbf{B} = \begin{pmatrix} \lambda_{23} \\ \lambda_{31} \\ \lambda_{12} \end{pmatrix}. \tag{G5}$$

Below, we first derive the integral form of Maxwell's equations, Eq. (G5a), from the integral form of the continuity equations of the curl vectors, Eq. (G2a), and from the integral form of the curl identities, Eq. (G3a). Then, we derive the differential form of Maxwell's equations, Eq. (G11b), from the differential form of the continuity equations of the curl vectors, Eq. (G2b), and the differential form of the curl identities, Eq. (G3b).

1. Integral form

Write the integral that expresses the space-time vector continuity of the curl vectors in Eq. (G2a) over a time slice dx_4 [Eq. (I22)] as

$$\oint \mathbf{\Lambda}_r\cdot\mathbf{n}dV = \left[\oint \mathbf{\Lambda}_r\cdot\mathbf{n}dS + \frac{\partial}{\partial x_4}\int \lambda_{r4}dV\right]dx_4. \tag{G6}$$

First, express the two integrals on the right side of Eq. (G6) as

$$\oint \mathbf{\Lambda}_r\cdot\mathbf{n}dS = \begin{cases} -i\oint \mathbf{B}\cdot d\mathbf{l}dx_r \ \ (r=1,2,3) \\ \oint \mathbf{E}\cdot\mathbf{n}dS \ \ (r=4) \end{cases},$$
$$\int \lambda_{r4}dV = \begin{cases} -\int \mathbf{E}\cdot\mathbf{n}dSdx_r \ \ (r=1,2,3) \\ 0 \ \ (r=4) \end{cases}. \tag{G7a,b}$$

Equation (G7a) follows from the mathematical manipulations given below:

$$\oint \mathbf{\Lambda}_1\cdot\mathbf{n}dS = \oint \begin{pmatrix} 0 \\ -iB_3 \\ iB_2 \end{pmatrix}\cdot\mathbf{n}dS$$
$$= i\left[-\left(B_3^+ - B_3^-\right)dx_3 + \left(B_2^+ - B_2^-\right)dx_2\right]dx_1$$
$$= -i\left(\oint_{\text{about 1}} \mathbf{B}\cdot d\mathbf{l}\right)dx_1,$$

$$\oint \mathbf{\Lambda}_2\cdot\mathbf{n}dS = \oint \begin{pmatrix} iB_3 \\ 0 \\ -iB_1 \end{pmatrix}\cdot\mathbf{n}dS$$
$$= i\left[\left(B_3^+ - B_3^-\right)dx_3 - \left(B_1^+ - B_1^-\right)dx_1\right]dx_2$$
$$= -i\left(\oint_{\text{about 2}} \mathbf{B}\cdot d\mathbf{l}\right)dx_2,$$

$$\oint \mathbf{\Lambda}_3\cdot\mathbf{n}dS = \oint \begin{pmatrix} -iB_2 \\ iB_1 \\ 0 \end{pmatrix}\cdot\mathbf{n}dS$$
$$= i\left[-\left(B_2^+ - B_2^-\right)dx_2 + \left(B_1^+ - B_1^-\right)dx_1\right]dx_3$$
$$= -i\left(\oint_{\text{about 3}} \mathbf{B}\cdot d\mathbf{l}\right)dx_3,$$

$$\oint \mathbf{\Lambda}_4\cdot\mathbf{n}dS = \oint \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}\cdot\mathbf{n}dS = \oint \mathbf{E}\cdot\mathbf{n}dS.$$

Equation (G7b) follows from the next set of mathematical manipulations:

$$\int \lambda_{14} dV = \left(\int_{2-3 \ \text{plane}} E_1 dS \right) dx_1 \quad \int \lambda_{24} dV = \left(\int_{3-1 \ \text{plane}} E_2 dS \right) dx_2,$$
$$\int \lambda_{34} dV = \left(\int_{1-2 \ \text{plane}} E_3 dS \right) dx_3 \quad \int \lambda_{44} dV = 0.$$

Next, let us consider the curl identities. Express Eq. (G3a) as

$$\oint_{(r,s,t)} \lambda_{rs} dx_r dx_s = \begin{cases} -\left[i \dfrac{\partial}{\partial x_4} \int_{(r,s,t)} \mathbf{B} \cdot \mathbf{n} dS + \oint_{\text{about } u} \mathbf{E} \cdot d\mathbf{l} \right] dx_4 & \text{for } u = 1,2,3 \\ i \oint \mathbf{B} \cdot \mathbf{n} dS \quad \text{for } u = 4 \end{cases}. \tag{G8}$$

Equation (G8) follows from the next set of mathematical manipulations:

$$\oint_{(2,4,3)} \lambda_{rs} dx_r dx_s = \left(E_2^+ - E_2^-\right) dx_2 \ dx_4 - \left(E_3^+ - E_3^-\right) dx_4 \ dx_3 - i\left(B_1^+ - B_1^-\right) dx_3 \ dx_2$$
$$= -\left[i \frac{\partial}{\partial x_4} \int_{(2,3)} \mathbf{B} \cdot \mathbf{n} dS + \oint_{\text{about } 1} \mathbf{E} \cdot d\mathbf{l} \right] dx_4,$$
$$\oint_{(3,4,1)} \lambda_{rs} dx_r dx_s = \left(E_3^+ - E_3^-\right) dx_3 \ dx_4 - \left(E_1^+ - E_1^-\right) dx_4 \ dx_1 - i\left(B_2^+ - B_2^-\right) dx_1 \ dx_3$$
$$= -\left[i \frac{\partial}{\partial x_4} \int_{(3,1)} \mathbf{B} \cdot \mathbf{n} dS + \oint_{\text{about } 2} \mathbf{E} \cdot d\mathbf{l} \right] dx_4,$$
$$\oint_{(4,2,1)} \lambda_{rs} dx_r dx_s = -\left(E_2^+ - E_2^-\right) dx_4 \ dx_2 - i\left(B_3^+ - B_3^-\right) dx_2 \ dx_1 + \left(E_1^+ - E_1^-\right) dx_1 \ dx_4$$
$$= -\left[i \frac{\partial}{\partial x_4} \int_{(1,2)} \mathbf{B} \cdot \mathbf{n} dS + \oint_{\text{about } 3} \mathbf{E} \cdot d\mathbf{l} \right] dx_4,$$
$$\oint_{(1,2,3)} \lambda_{rs} dx_r dx_s = i\left(B_3^+ - B_3^-\right) dx_1 \ dx_2 + i\left(B_1^+ - B_1^-\right) dx_2 \ dx_3 + i\left(B_2^+ - B_2^-\right) dx_3 \ dx_1$$
$$= i \oint \mathbf{B} \cdot \mathbf{n} dS.$$

Write the left side and the right side of Eq. (G2a) for ($r = 1, 2, 3$) as

$$4\pi\alpha_r \int \delta_3 dE = 4\pi \int \alpha e_r \delta_3 dV dx_4 = \frac{4\pi}{ic} \int j_r \delta_3 dV dx_4 = -i\left(\frac{4\pi}{c} \int \mathbf{j} \cdot \mathbf{n} dS \right) dx_r dx_4$$
$$\oint \mathbf{\Lambda}_r \cdot \mathbf{n} dV = \left[-i \oint \mathbf{B} \cdot d\mathbf{l} dx_r \ - \frac{\partial}{\partial x_4} \left(\int \mathbf{E} \cdot \mathbf{n} dS dx_r \ \right) \right] dx_4 = -i\left[\oint \mathbf{B} \cdot d\mathbf{l} - \frac{1}{c}\frac{\partial}{\partial t} \int \mathbf{E} \cdot \mathbf{n} dS \right] dx_r \ dx_4,$$

where $\alpha_r = \alpha e_r = q v_r / ic \ = j_r / ic$ for $(r = 1,2,3)$ and $\alpha_4 = q$. Equating the left side and the right side yields Maxwell equation (G4a). Next, write the left side and the right side of Eq. (G2a) for ($r = 4$) as

$$4\pi\alpha_4 \int \delta_3 dE = 4\pi\alpha \int \delta_3 dV dx_4 = 4\pi q dx_4 \quad \oint \mathbf{\Lambda}_r \cdot \mathbf{n} dV = \left[\oint \mathbf{E} \cdot \mathbf{n} dS \right] dx_4.$$

Equating the left side and the right side yields Maxwell equation (G4b). Finally, notice that Eq. (*G*8) for ($u = 1, 2, 3$) is Maxwell equation (G4c) and that Eq. (G8) for ($u = 4$) is Maxwell equation (G4d).

2. Differential form

Next, we independently derive the differential form of Maxwell's equations, Eq. (*G*4), from Eqs. (G2b) and (G3b). From Eqs. (G5) and (G2b)

$$4\pi\frac{j_1}{ic}\delta_3=\frac{\partial(0)}{\partial x_1}+\frac{\partial(-iB_3)}{\partial x_2}+\frac{\partial(iB_2)}{\partial x_3}+\frac{\partial(-E_1)}{\partial x_4},$$
$$4\pi\frac{j_2}{ic}\delta_3=\frac{\partial(iB_3)}{\partial x_1}+\frac{\partial(0)}{\partial x_2}+\frac{\partial(-iB_1)}{\partial x_3}+\frac{\partial(-E_2)}{\partial x_4},$$
$$4\pi\frac{j_3}{ic}\delta_3=\frac{\partial(-iB_2)}{\partial x_1}+\frac{\partial(iB_1)}{\partial x_2}+\frac{\partial(0)}{\partial x_3}+\frac{\partial(-E_3)}{\partial x_4},$$
$$4\pi q\delta_3=\frac{\partial(E_1)}{\partial x_1}+\frac{\partial(E_2)}{\partial x_2}+\frac{\partial(E_3)}{\partial x_3}+\frac{\partial(0)}{\partial x_4}.$$

In vector notation,

$$4\pi\frac{\mathbf{j}}{ic}\delta_3=-i\nabla\times\mathbf{B}-\frac{\partial\mathbf{E}}{\partial x_4}\quad 4\pi q\delta_3=\nabla\cdot\mathbf{E}\ . \tag{G9a,b}$$

Next, consider the differential form of the curl identities, Eq. (G3b). From Eq. (G5)

$$0=\frac{\partial(iB_3)}{\partial x_3}+\frac{\partial(iB_1)}{\partial x_1}+\frac{\partial(iB_2)}{\partial x_2}$$
$$0=\frac{\partial(E_2)}{\partial x_3}+\frac{\partial(-E_3)}{\partial x_2}+\frac{\partial(-iB_1)}{\partial x_4},$$
$$0=\frac{\partial(E_3)}{\partial x_1}+\frac{\partial(-E_1)}{\partial x_3}+\frac{\partial(-iB_2)}{\partial x_4}$$
$$0=\frac{\partial(-E_2)}{\partial x_1}+\frac{\partial(-iB_3)}{\partial x_4}+\frac{\partial(E_1)}{\partial x_2}.$$

In vector notation,

$$0=i\nabla\cdot\mathbf{B}\qquad 0=-\nabla\times\mathbf{E}-i\frac{\partial\mathbf{B}}{\partial x_4}. \tag{G10a,b}$$

Equation (G9a) yields the first Maxwell's equation, Eq. (G4a). Equation (*G*9b) yields the second Maxwell's equation, Eq. (G4b). Equation (G10a) yields the third Maxwell's equation, Eq. (G4c). Equation (G10b) yields the fourth Maxwell's equation, Eq. (G4d).

APPENDIX H: GRAVITATIONAL EQUATIONS FROM GR

Below we obtain an exact solution to the one-body gravitational problem beginning from the Schwarzschild metric. Note that a perturbation solution is also available.[8] Begin with the Schwarzschild metric, written as[h)]

$$\begin{aligned}ds^2&=c^2d\tau^2\\&=c^2\left(1-\frac{r_s}{r}\right)dt^2-\frac{1}{1-\frac{r_s}{r}}dr^2-\left(d\theta^2+\sin^2\theta\ d\varphi^2\right).\end{aligned} \tag{H1}$$

In Eq. (H1), τ is proper time and $r_s=2GM/c^2$ is the Schwarzschild radius. For the one-body gravitational problem, let $\theta=\pi/2$ and $d/d\tau=(\dot{})$. From Eq. (H1),

$$c^2=f(r,\dot r,\ \dot t,\dot\varphi)=c^2\left(1-\frac{r_s}{r}\right)\dot t^2-\frac{1}{1-\frac{r_s}{r}}\dot r^2-r^2\dot\varphi^2. \tag{H2}$$

[h)]David R. Williams, NASA Goddard Space Flight Center, https://nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfact.html

Integrating Eq. (H2) by parts, recognizing that r, t, and φ are independent

$$\begin{aligned}0&=\int df\\&=\int\left(\frac{\partial f}{\partial r}dr+\frac{\partial f}{\partial\dot r}d\dot r+\frac{\partial f}{\partial\dot t}d\dot t+\frac{\partial f}{\partial\dot\varphi}d\dot\varphi\right),\\&=\int\left[\left(\frac{\partial f}{\partial r}-\frac{d}{d\tau}\left(\frac{\partial f}{\partial\dot r}\right)\right)dr-\frac{d}{d\tau}\left(\frac{\partial f}{\partial\dot t}\right)dt-\frac{d}{d\tau}\left(\frac{\partial f}{\partial\dot\varphi}\right)d\varphi\right],\end{aligned}$$

from which

$$0=\frac{\partial f}{\partial r}-\frac{d}{d\tau}\left(\frac{\partial f}{\partial\dot r}\right),\quad 0=\frac{d}{d\tau}\left(\frac{\partial f}{\partial\dot t}\right),\quad 0=\frac{d}{d\tau}\left(\frac{\partial f}{\partial\dot\varphi}\right). \tag{H3}$$

From Eqs. (H2) and (H3),

$$\frac{\partial f}{\partial\dot t}=2\left(1-\frac{r_s}{r}\right)\dot t=\frac{2E}{mc^2}\qquad\frac{\partial f}{\partial\dot\varphi}=2r^2\dot\varphi=\frac{2H_C}{\mu}$$
$$\mu=\frac{mM}{m+M},$$

and so

$$\begin{aligned}\frac{m}{2}\dot r^2&=\left(\frac{E^2}{2mc^2}-\frac{mc^2}{2}\right)-\left(\frac{H_C}{\mu}\right)^2\frac{m}{2r^2}+\frac{GMm}{r}\\&\quad+\left(\frac{H_C}{c\mu}\right)^2\frac{GMm}{r^3}.\end{aligned} \tag{H4}$$

Equation (H4) expresses conservation of energy

$$C=(T_r+T_\varphi)+(V+V_{GR}), \tag{H5}$$

in which $T_r=(m/2)\dot r^2$ and $T_\varphi=(H_C/\mu)^2m/2r^2$ are the radial and tangential components of kinetic energy and $V=-GMm/r$ and $V_{GR}=-(H_C/c\mu)^2GMm/r^3$ are the Newtonian potential and the effective radial potential. The effective radial potential V_{GR} is the term in the potential energy that GR introduces. From Eq. (H4),

$$V+V_{GR}=-\frac{GMm}{r}-\left(\frac{H_C}{c\mu}\right)^2\frac{GMm}{r^3}\ . \tag{H6}$$

Equation (H6) is the gravitational potential given in Table Ib, in which $H/m=H_C/\mu$.

APPENDIX I. ELEMENTS OF 4D VECTOR FIELDS

1. Space

The geometry of space: One deduces the geometry of an n-dimensional space from the algebraic relations of its associated n-dimensional cubes. In 3D space, the 3-cube (ordinary cube) has three pairs of positive and negative faces, each representing a 2-cube (square). In 4D space, the 4-cube has four pairs of positive and negative faces, each representing a 3-cube.

The right-hand rule: In 3D space, one customarily sets up the coordinates in a right-handed order to ensure consistency in subsequently defined operations. In 4D space, the right-handedness of the four possible triads is not obvious. For example, is (1, 4, 3) or (1, 3, 4) right-handed when 4 is associated with geometric time? One determines the correct "right-handedness" from the assembly of the 3-cubes into a 4-cube. Figure 6 shows an easy way to remember the resulting right-hand rules. As shown, one starts with a tetrahedron. The vertices of its base are labeled 1, 2, and 3 in the right-handed order, and its top vertex is labeled as 4. Next, one opens the tetrahedron like the pedals of a flower and lays it flat. The right-handed orders of the triads express the right-hand rules.

As shown, the triads (1, 2, 3), (2, 4, 3), (3, 4, 1), and (4, 2, 1) are each right-handed and so too are the quads (1, 2, 3, 4), (2, 4, 3, 1), (3, 4, 1, 2), and (4, 2, 1, 3). Switching any two adjacent indices converts a right-handed space into a left-handed space and vice-versa. Mathematically, one can express the right-hand rules for a 4D space by the permutation symbol ε_{rstu}. One defines it such that $\varepsilon_{rstu} = 0$ when any two indices repeat, such that $\varepsilon_{rstu} = 1$ when the indices are in the right-handed order, and such that $\varepsilon_{rstu} = -1$ when the indices are in the left-handed order.

2. Vectors

The vector rules: Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ denote n-dimensional vectors and let a and b denote numbers (scalars). Write each vector as a column of n numbers $\mathbf{x} = [x_1\ x_2\ x_3\ x_4]^T$ and define vector addition and multiplication by a scalar as follows: Add two vectors by adding together the numbers in their corresponding entries. Multiply the vector by a scalar by multiplying each entry by the scalar. Let n-dimensional vectors satisfy the following two commutative rules, two associative rules, and distributive rule:

$$\begin{gathered}\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x} \quad a\mathbf{x}=\mathbf{x}a \\ (\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z}) \quad (\mathbf{x}a)b=\mathbf{x}(ab). \\ a(\mathbf{x}+\mathbf{y})=a\mathbf{x}+b\mathbf{y}\end{gathered} \tag{I1a-e}$$

Next, define the dot product of the two vectors $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_1y_1 + \cdots + x_ny_n. \tag{I2}$$

Using the dot product operation, the magnitude of a vector $\mathbf{x}$ and the component x_y of a vector $\mathbf{x}$ in the direction of the vector $\mathbf{y}$ are

$$|\mathbf{x}| = \sqrt{\mathbf{x}\cdot\mathbf{x}}, \quad x_y = \mathbf{x}\cdot\frac{\mathbf{x}}{|\mathbf{x}|}. \tag{I3a,b}$$

The path: Next, let $\mathbf{x}$ represent a position vector over a path. The path increment is ds. The unit vector tangent to the path is the path derivative of the position vector, and the curvature vector is the path derivative of the unit vector tangent to the path, written

$$\mathbf{e} = \frac{d\mathbf{x}}{ds} \quad \mathbf{k} = \frac{d\mathbf{e}}{ds}. \tag{I4a,b}$$

Notice that the curvature vector and the unit vector are perpendicular, because $0 = \frac{d}{ds}(1) = \frac{d}{ds}(\mathbf{e}\cdot\mathbf{e}) = 2\mathbf{k}\cdot\mathbf{e}$.

The perpendicular vector: Next, let us define the perpendicular vectors for 2D, 3D, and 4D spaces. In a 2D space, let the 2D vector $\mathbf{w}$ be perpendicular to $\mathbf{x}$. In a 3D space, let the 3D vector $\mathbf{w}$ be perpendicular to $\mathbf{x}$ and $\mathbf{y}$, and in a 4D space let the 4D vector $\mathbf{w}$ be perpendicular to $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. One expresses the formulas for the perpendicular vectors in terms of the permutation symbols as follows:

$$\begin{aligned}&\text{For 2D spaces: } w_r = \sum_{s=1}^{2}\varepsilon_{rs}x_s, \\ &\text{For 3D spaces: } w_r = \sum_{s=1}^{3}\sum_{t=1}^{3}\varepsilon_{rst}x_sy_t, \\ &\text{For 4D spaces: } w_r = \sum_{s=1}^{4}\sum_{t=1}^{4}\sum_{u=1}^{4}\varepsilon_{rstu}x_sy_tz_u.\end{aligned} \tag{I5a-c}$$

Equation (I5) expresses the components of the perpendicular vectors in an index notation. In the 2D case, one more commonly expresses Eq. (I5a) as $w = ix$, in which w and x are complex numbers and where i is the imaginary number, recognizing that a complex number is equivalent to a 2D vector and that the imaginary number i represents a 90° counterclockwise rotation of the 2D vector. In the 3D case, one more commonly expresses Eq. (I5b) as the cross product $\mathbf{w} = \mathbf{x}\times\mathbf{y}$.

The 4D space-time vector: Let us now partition the 4D position vector $\mathbf{x}$ into a 3D spatial position vector $\boldsymbol{x}$ and a temporal position vector $x_4 = c_0t$. The partitioned position vector, unit vector, and curvature vector are

$$\begin{aligned}&\mathbf{x} = \begin{pmatrix}\boldsymbol{x}\\ c_0t\end{pmatrix}, \quad \mathbf{e} = \beta_0\begin{pmatrix}\boldsymbol{v}/c_0\\ 1\end{pmatrix}, \\ &\mathbf{k} = \frac{\beta_0^2}{c_0^2}\left[\begin{pmatrix}\boldsymbol{a}\\ 0\end{pmatrix} - va\frac{\beta_0^2}{c_0}\begin{pmatrix}\boldsymbol{v}/c_0\\ 1\end{pmatrix}\right],\end{aligned} \tag{I6a-c}$$

where

$$\boldsymbol{x} = \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} \quad\text{and}\quad \beta_0 = \frac{1}{\sqrt{1+\left(\frac{v}{c_0^2}\right)^2}}.$$

Under the nonrelativistic speed assumption, for which $v \ll c_0$, the unit vector and the curvature vector are approximated by (see Fig. 7)

$$\mathbf{e} \cong \frac{1}{c_0}\begin{pmatrix}\mathbf{v}\\ c_0\end{pmatrix}, \quad \mathbf{k} \cong \frac{1}{c_0^2}\begin{pmatrix}\mathbf{a}\\ -\dfrac{\mathbf{a}\cdot\mathbf{v}}{c_0}\end{pmatrix}. \tag{I7a,b}$$

The speed of light: In space-time physics, one produces geometric time x_4 by rotating the radially directed propagation length ct of an electromagnetic wave to a geometric time axis. This is expressed mathematically by the equation $x_4 = ict$ where, again, one recognizes that i is a 90° rotation

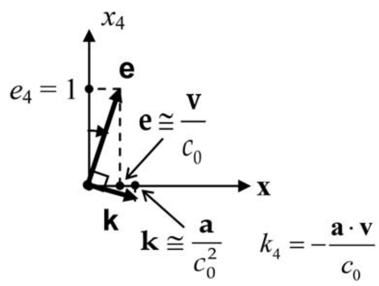

FIG. 7. **e** and **k** under the nonrelativistic speed assumption.

of a 2D vector (in radial-time). Comparing $x_4 = c_0 t$ above Eq. (I6) and $x_4 = ict$ above, one finds that $c_0 = ic$. From Eq. (I6),

$$\mathbf{e} = \beta \begin{pmatrix} \mathbf{v}/ic \\ 1 \end{pmatrix}, \quad \mathbf{k} = -\frac{\beta^2}{c^2}\left[\begin{pmatrix} \mathbf{a} \\ 0 \end{pmatrix} - va\frac{\beta^2}{ic}\begin{pmatrix} \mathbf{v}/ic \\ 1 \end{pmatrix}\right], \tag{I8a,b}$$

where $\beta = 1/\sqrt{1-\left(\frac{v}{c}\right)^2}$. From Eq. (I8), under the nonrelativistic speed approximation

$$\mathrm{e} = \frac{1}{ic}\begin{pmatrix} \mathbf{v} \\ ic \end{pmatrix}, \quad \mathrm{k} = -\frac{1}{c^2}\begin{pmatrix} \mathbf{a} \\ \dfrac{i\mathbf{a}\cdot\mathbf{v}}{c} \end{pmatrix}. \tag{I8c}$$

Summary: In 3D-vector analysis, space-time paths are kinematic, expressed in terms of a position vector, a velocity vector, and an acceleration vector. In 4D-vector analysis, in particular, in space-time analysis, space-time paths are geometric, expressed in terms of a position vector, a unit vector tangent to the path, and a curvature vector of the path

$$3D\text{ analysis}: \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{v} = \frac{d\mathbf{x}}{dt} \quad \mathbf{a} = \frac{d\mathbf{v}}{dt},$$

$$4D\text{ analysis}: \quad \mathbf{x} = \begin{pmatrix} \mathbf{x} \\ x_4 \end{pmatrix} \quad \mathbf{e} = \frac{d\mathrm{x}}{ds} \quad \mathbf{k} = \frac{d\mathbf{e}}{ds}. \tag{I9a,b}$$

3. Vector fields

Let us now examine continuous vector fields in 2D, 3D, and 4D spaces. In particular, we present the integral theorems and the differential theorems that they obey. We will do this systematically, by distinguishing between integrals of vector components along lines of integration and integrals of vector components that are perpendicular to lines of integration, called longitudinal theorems and transverse theorems, respectively. The longitudinal theorems turn integrals around closed lines into integrals in open surfaces, integrals around closed surfaces into integrals in open volumes, and integrals around closed volumes into integrals in open events. The transverse theorems are similar. When performing the manipulations, we will denote a differential line element by $dx_1^{(1)}$, a differential surface element by $dx_1^{(2)}dx_2^{(2)}$, a differential volume element by $dx_1^{(3)}dx_2^{(3)}dx_3^{(3)}$, and a differential event by $dx_1^{(4)}dx_2^{(4)}dx_3^{(4)}dx_4^{(4)}$.

Longitudinal theorems: Denote differential circulation by $d\Gamma = A_1^{(1)}dx_1^{(1)}$, where ${A_1}^{(1)}$ is the component of **A** in the direction of the differential line element ${dx_1}^{(1)}$. Also, denote differential curl by $d\Lambda = \lambda_{12}^{(2)}dx_1^{(2)}dx_2^{(2)}$, where $\lambda_{12}^{(2)} = (\partial A_2^{(2)}/\partial x_1^{(2)}) - (\partial A_1^{(2)}/\partial x_2^{(2)})$ denotes curl density and denote differential coil by $d\Omega = \omega^{(3)}dx_1^{(3)}dx_2^{(3)}dx_3^{(3)}$ where $\omega^{(3)} = (\partial\lambda_{12}^{(3)}/\partial x_3^{(3)}) + (\partial\lambda_{23}^{(3)}/\partial x_1^{(3)}) + (\partial\lambda_{31}^{(3)}/\partial x_2^{(3)})$ denotes coil density.

First, consider an open finite surface divided into differential surface elements. Differential circulations along differential line elements from adjacent differential surfaces cancel. The only differential circulations that are left are along the face of the finite surface. It follows that the circulation around the boundary of the finite surface is equal to the curl inside the surface. It also follows that both are equal to zero if the surface is closed. Next, consider a finite volume divided into differential volume elements. Differential curls along differential surface elements from adjacent differential volumes cancel. The only differential curls that are left are along the face of the finite volume. It follows that the curl around the face of the finite volume is equal to the coil inside the volume. It also follows that both are equal to zero if the volume is closed and that the coil density inside the closed volume is identically equal to zero. The following lists the longitudinal theorems:

$$\begin{array}{lll} & \text{circulation} & \Gamma = \oint A_1^{(1)}dx_1^{(1)}, \\ \Gamma_{closed} = \Lambda_{open} & & \\ \Lambda_{closed} = \Omega = 0 & \text{curl} & \Lambda = \oint \lambda_{12}^{(3)}dx_1^{(2)}dx_2^{(2)}, \\ & \text{coil} & \Omega = \oint \omega^{(3)}dx_1^{(3)}dx_2^{(3)}dx_3^{(3)}. \end{array} \tag{I10}$$

One can also show by simple substitution that

$$0 = \omega^{(3)} = \frac{\partial\lambda_{12}^{(3)}}{\partial x_3^{(3)}} + \frac{\partial\lambda_{23}^{(3)}}{\partial x_1^{(3)}} + \frac{\partial\lambda_{31}^{(3)}}{\partial x_2^{(3)}}. \tag{I11}$$

Transverse theorems: First, for 2D spaces, denote differential stream by $d\Delta = A_2^{(1)}dx_1^{(1)}$, where ${A_2}^{(1)}$ is the component of **A** perpendicular to the differential line element ${dx_1}^{(1)}$ and denote differential leak by $dX = \xi dx_1^{(2)}dx_2^{(2)}$, where ξ is leak density. For 3D spaces, denote differential flux by $d\Phi = A_3^{(2)}dx_1^{(2)}dx_2^{(2)}$, where ${A_3}^{(2)}$ is the component of **A** perpendicular to the differential surface element ${dx_1}^{(2)}{dx_2}^{(2)}$ and denote differential divergence by $dY = \psi dx_1^{(3)}dx_2^{(3)}dx_3^{(3)}$, where ψ is divergence density. For 4D spaces, denote differential strength by $d\Psi = A_4^{(3)}dx_1^{(3)}dx_2^{(3)}dx_3^{(3)}$, where ${A_4}^{(3)}$ is the component of **A** perpendicular to the differential volume element $dx_1{}^{(3)}dx_2{}^{(3)}dx_3{}^{(3)}$ and denote differential dilatation by $dZ = \zeta dx_1^{(4)}dx_2^{(4)}dx_3^{(4)}dx_4^{(4)}$ where ζ is dilatation density.

For 2D spaces, consider an open finite surface divided into differential surface elements. Differential streams along differential line elements from adjacent differential surfaces cancel. The only differential streams that are left are along the face of the finite surface. It follows that the stream around the boundary of the finite surface is equal to the leak inside the surface. For 3D spaces, consider an open finite volume divided into differential volume elements. Differential fluxes on differential surface elements from adjacent differential volumes cancel. The only differential fluxes that are left are along the face of the finite volume. It follows that the flux around the boundary of the finite volume is equal to the divergence inside the volume. For 4D spaces, consider an open finite event divided into differential event elements. Differential strengths on differential volume elements from adjacent differential events cancel. The only differential strengths that are left are along the face of the finite event. It follows that the strength around the boundary of the finite event is equal to the dilatation inside the event. The following summarizes these results:

$$\Delta = X \quad \begin{array}{l} \text{stream } \Delta = \oint A_2^{(1)} dx_1^{(1)} \\ \text{leak } X = \int \xi dx_1^{(2)} dx_2^{(2)} \end{array},$$

$$\Phi = Y \quad \begin{array}{l} \text{flux } \Phi = \oint A_3^{(2)} dx_1^{(2)} dx_2^{(2)} \\ \text{divergence } Y = \int \psi dx_1^{(3)} dx_2^{(3)} dx_3^{(3)} \end{array}, \tag{I12}$$

$$\Psi = Z \quad \begin{array}{l} \text{strength } \Psi = \oint A_4^{(3)} dx_1^{(3)} dx_2^{(3)} dx_3^{(3)} \\ \text{dilatation Z} = \int \zeta dx_1^{(4)} dx_2^{(4)} dx_3^{(4)} dx_4^{(4)} \end{array},$$

where

$$\xi = \frac{\partial A_1^{(2)}}{\partial x_1^{(2)}} + \frac{\partial A_2^{(2)}}{\partial x_2^{(2)}} \qquad \psi = \frac{\partial A_1^{(3)}}{\partial x_1^{(3)}} + \frac{\partial A_2^{(3)}}{\partial x_2^{(3)}} + \frac{\partial A_3^{(3)}}{\partial x_3^{(3)}}$$

$$\zeta = \frac{\partial A_1^{(4)}}{\partial x_1^{(4)}} + \frac{\partial A_2^{(4)}}{\partial x_2^{(4)}} + \frac{\partial A_3^{(4)}}{\partial x_3^{(4)}} + \frac{\partial A_4^{(4)}}{\partial x_4^{(4)}}. \tag{I13}$$

4. Vector continuity

Appendix Section I 3 enforced continuity of the individual components of a vector field, that is, scalar continuity. This section considers, in addition to scalar continuity, continuity of the vector field as a whole. Toward this end, one defines the concept of a field line.

The field line: The number of field lines that passes through a differential face of an n-dimensional body is (see Fig. 8)

$$dN = \boldsymbol{A} dF_\perp = \mathrm{A}_\perp dF = \mathrm{A} \cdot \mathrm{n} dF, \tag{I14}$$

where dF is a differential face, **n** is the outward normal to the differential face, and where $(\,)_\perp$ means perpendicular component.

Vector continuity requires the number of field lines that leaves an n-dimensional domain (through its faces) to be equal to zero, written

$$0 = \oint dN = \oint \mathbf{A} \cdot \mathbf{n} \; dF. \tag{I15}$$

Note from Eq. (I14) that a field line that enters a domain is equal to the negative of a field line that leaves it.

Vector continuity of A: By comparing the transverse integral theorems developed in Appendix Section I 3 with Eq. (I15), it follows: for 2D spaces that the stream and leak density are zero, for 3D spaces that the flux and divergence density are zero, and for 4D spaces that the strength and dilatation density are zero

$$\begin{array}{lll} & \text{integral forms} & \text{differential forms} \\ \text{For 2D spaces: leak} & 0 = \oint \mathbf{A} \cdot \mathbf{n} \; dl & 0 = \dfrac{\partial A_1}{\partial x_1} + \dfrac{\partial A_2}{\partial x_2} \quad (or\;\; 0 = \vee \cdot \mathbf{A}), \\ \text{For 3D spaces: divergence} & 0 = \oint \mathbf{A} \cdot \mathbf{n} \; dS & 0 = \dfrac{\partial A_1}{\partial x_1} + \dfrac{\partial A_2}{\partial x_2} + \dfrac{\partial A_3}{\partial x_3} \quad (or\;\; 0 = \nabla \cdot \mathbf{A}), \\ \text{For 4D spaces: dilatation} & 0 = \oint \mathbf{A} \cdot \mathbf{n} \; dV & 0 = \dfrac{\partial A_1}{\partial x_1} + \dfrac{\partial A_2}{\partial x_2} + \dfrac{\partial A_3}{\partial x_3} + \dfrac{\partial A_4}{\partial x_4} \quad (or\;\; 0 = \Box \cdot \mathbf{A}). \end{array} \tag{I16a-c}$$

Vector continuity of the gradient vectors: Next, write the gradient vectors of the components of the vector fields as

$$\begin{aligned}
&\text{For 2D spaces:} \quad \mathbf{C}_i = \begin{pmatrix} \frac{\partial A_i}{\partial x_1} & \frac{\partial A_i}{\partial x_2} \end{pmatrix}^T \quad \mathbf{C}_i = \vee A_i, \\
&\text{For 3D spaces:} \quad \mathbf{C}_i = \begin{pmatrix} \frac{\partial A_i}{\partial x_1} & \frac{\partial A_i}{\partial x_2} & \frac{\partial A_i}{\partial x_3} \end{pmatrix}^T \quad \mathbf{C}_i = \nabla A_i \ , \\
&\text{For 4D spaces:} \quad \mathbf{C}_i = \begin{pmatrix} \frac{\partial A_i}{\partial x_1} & \frac{\partial A_i}{\partial x_2} & \frac{\partial A_i}{\partial x_3} & \frac{\partial A_i}{\partial x_4} \end{pmatrix}^T \quad \mathbf{C}_i = \Box A_i.
\end{aligned} \tag{I17a-c}$$

Write the enforcement of vector continuity of the gradient vectors in Eqs. (I17a–c) as

$$\begin{aligned}
&\text{For 2D spaces:} \quad \begin{matrix} 0 = \oint \vee A_i \cdot \mathbf{n}\ dl \\ (i = 1, 2) \end{matrix} \quad \begin{matrix} 0 = \frac{\partial^2 A_i}{\partial x_1^2} + \frac{\partial^2 A_i}{\partial x_2^2} \\ (or\ \ 0 = \ \vee \cdot \vee A_i) \end{matrix}, \\
&\text{For 3D spaces:} \quad \begin{matrix} 0 = \oint \nabla A_i \cdot \mathbf{n}\ ds \\ (i = 1, 2, 3) \end{matrix} \quad \begin{matrix} 0 = \frac{\partial^2 A_i}{\partial x_1^2} + \frac{\partial^2 A_i}{\partial x_2^2} + \frac{\partial^2 A_i}{\partial x_3^2} \\ (or\ \ 0 = \ \nabla \cdot \nabla A_i) \end{matrix}, \\
&\text{For 4D spaces:} \quad \begin{matrix} 0 = \oint \Box A_i \cdot \mathbf{n}\ dV \\ (i = 1, 2, 3, 4) \end{matrix} \quad \begin{matrix} 0 = \frac{\partial^2 A_i}{\partial x_1^2} + \frac{\partial^2 A_i}{\partial x_2^2} + \frac{\partial^2 A_i}{\partial x_3^2} + \frac{\partial^2 A_i}{\partial x_4^2} \\ (or\ \ 0 = \ \Box \cdot \ \Box A_i) \end{matrix}.
\end{aligned} \tag{I18a-c}$$

Equation (I18a) are also called potential equations and, by letting $x_4 = ict$, Eq. (I18c) is identically the wave equation $0 = \frac{\partial^2 A_i}{\partial x_1^2} + \frac{\partial^2 A_i}{\partial x_2^2} + \frac{\partial^2 A_i}{\partial x_3^2} - \frac{1}{c^2}\frac{\partial^2 A_i}{\partial t^2}$. The enforcement of the vector continuity equations, along with the appropriate boundary conditions, yields a solution that is unique up to an additive constant (Appendix C).

Vector continuity over a time increment: In the following, we consider the integral form of vector continuity in a 4D space, Eq. (I16c), over a time increment. The domain is now finite in the spatial dimensions and infinitesimal in the temporal dimension. It is an infinitesimally thin slice in time of a finite spatial domain. Over this time slice, we express Eq. (I16c) as the sum of two terms, one associated with flow lines leaving a spatial volume, and the other associated with a time rate of generation of field lines in the spatial volume. This will yield the classical 3D form of conservation.

The faces of the time slice consist of four pairs of positive and negative 3D cubes, each pair having the differential volume elements

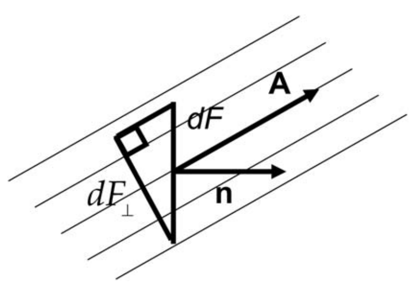

FIG. 8. Field lines.

$$\begin{aligned}
&dV_1 = dx_3 dx_2 dx_4 = dS_{23} dx_4 \quad dV_2 = dx_1 dx_3 dx_4 = dS_{31} dx_4, \\
&dV_3 = dx_2 dx_1 dx_4 = dS_{12} dx_4 \quad dV_4 = dx_1 dx_2 dx_3 = dV.
\end{aligned} \tag{I19a-d}$$

Partition the vector field as follows:

$$\begin{aligned}
&\mathbf{A} = \begin{pmatrix} \mathbf{A} \\ A_4 \end{pmatrix} = A\mathbf{e} = A \begin{pmatrix} \mathbf{v} \\ c_0 \end{pmatrix} \frac{1}{\sqrt{v^2 + c_0^2}} \\
&\mathbf{A} \cdot \mathbf{n} = A\mathbf{v} \cdot \mathbf{n} \frac{1}{\sqrt{v^2 + c_0^2}} = \frac{1}{c_0} A_4 \mathbf{v} \cdot \mathbf{n}.
\end{aligned} \tag{I20a,b}$$

Substitute Eqs. (I19) and (I20) into Eq. (I16c) to get

$$\oint \mathbf{A} \cdot \mathbf{n}\ dV = \left[\oint \mathbf{A} \cdot \mathbf{n}\ dS + \frac{\partial}{\partial x_4} \int A_4 dV \right] dx_4. \tag{I21}$$

In Eq. (I21), the flow out of the eight 3D cubes that make up the faces of a 4D cube now flows out of the two integrals $\oint \mathbf{A} \cdot \mathbf{n}\ dS$ and $\int A_4 dV$. It flows out of the six spatial faces of $\oint \mathbf{A} \cdot \mathbf{n}\ dS$ and the two temporal faces of $\int A_4 dV$. Letting $\partial/\partial x_4 = 1/c_0\, \partial/\partial t$, from Eq. (I21), one expresses the satisfaction of vector continuity over a time slice as

$$\begin{aligned}
&0 = \oint \mathbf{A} \cdot \mathbf{n}\ dS + \frac{\partial}{\partial x_4} \int A_4 dV, \\
&\text{or} \\
&0 = \oint A_4 \mathbf{v} \cdot \mathbf{n}\ dS + \frac{\partial}{\partial t} \int A_4 dV.
\end{aligned} \tag{I22a,b}$$

One conventionally refers to Eqs. (I22) as a conservation law.

5. The point singularity

Assume that an n-dimensional region contains a point singularity. To assist with the treatment of the singularity, we introduce the n-dimensional Kronecker-delta function and the Dirac-delta function, defined as follows:

$$\delta_{rs} = \begin{cases} 1 & r = s \\ 0 & r \neq s \end{cases} \quad (r, s = 1, 2, \ldots, n), \tag{I23}$$

$$\delta_n(\mathbf{x}) = 0 \text{ for } \mathbf{x} \neq \mathbf{0}, \quad \int_D \delta_n(\mathbf{x}) dD_n = \begin{cases} 1 & D_n \notin \mathbf{0} \\ 0 & D_n \in \mathbf{0} \end{cases}. \tag{I24}$$

The Dirac-delta function is a unit impulse at the origin. The value of the function approaches infinity as the size of the infinitesimal region surrounding it approaches zero in reciprocal proportion, such that the integral of the function over any domain containing $\mathbf{0}$ is equal to one. The Kronecker-delta function and the Dirac-delta function make it possible to, respectively, evaluate an element of a vector, and to evaluate an integrand of a function at a point. For example, the sum $f_r = \sum_{s=1}^{n} \delta_{rs} f_s$ selects the rth component of the vector $\mathbf{f}$ and the integral $f(\mathbf{0}) = \int_D f(\mathbf{x}) \delta_n(\mathbf{x}) dD_n$ evaluates the integrand at $\mathbf{x} = \mathbf{0}$.

Now consider a scalar field A that is continuous in a 3D region except at its singularity. First, examine the flux that the 3D vector field $\partial A/\partial \mathbf{x}$ produces. We say that the flux that leaves the surface of a volume that contains the singularity is equal to $4\pi\Theta$. The flux that leaves the surface of any volume that does not contain the singularity is equal to zero. Using the 3D Dirac-delta function, we can write both of these facts together as

$$\oint \frac{\partial A}{\partial \mathbf{x}} \cdot \mathbf{n} \; dS = 4\pi\Theta \int \delta_3 dV. \tag{I25}$$

Next, consider the differential form of Eq. (I25), given in Eq. (I18b) at any point except at the point singularity. Using the 3D Dirac-delta function, we can make Eq. (I18b) apply to the point singularity, too, by writing it as

$$4\pi\Theta\delta_3 = \frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_2^2} + \frac{\partial^2 A}{\partial x_3^2}. \tag{I26}$$

The left side of Eq. (I26) is equal to zero for any point that is not at the singularity. The notation implies over the volumetric neighborhood of the singularity that one sets the volume integral on the right side to the surface integral given in Eq. (I25) to yield a flux $4\pi\Theta$.

Finally, let us consider Eq. (I21) for $\partial A/\partial \mathbf{x}$. We now regard the 3D vector field $\partial A/\partial \mathbf{x}$ as the spatial part of the 4D vector field $\partial A/\partial \mathbf{x}$. Using the 3D Dirac-delta function, we can write

$$\oint \frac{\partial A}{\partial \mathbf{x}} \cdot \mathbf{n} \; dV = \left(4\pi\Theta \int \delta_3 dV\right) dx_4 = 4\pi\Theta \int \delta_3 dE. \tag{I27}$$

To accommodate the singularity, we write the differential form given in Eq. (I18c) as

$$4\pi\Theta\delta_3 = \frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_2^2} + \frac{\partial^2 A}{\partial x_3^2} + \frac{\partial^2 A}{\partial x_4^2}. \tag{I28}$$

Doing the same for the components $\partial A_r/\partial \mathrm{x}$ $(r = 1, 2, 3, 4)$, accounting for the singularity, we write

integral form — differential form

$$4\pi\Theta_r \int \delta_3 dE = \oint \frac{\partial A_r}{\partial \mathbf{x}} \cdot \mathbf{n} \; dV \quad (r = 1, 2, 3, 4) \qquad 4\pi\Theta_r\delta_3 = \frac{\partial^2 A_r}{\partial x_1^2} + \frac{\partial^2 A_r}{\partial x_2^2} + \frac{\partial^2 A_r}{\partial x_3^2} + \frac{\partial^2 A_r}{\partial x_4^2} \quad (\text{or } 4\pi\Theta_r\delta_3 = \Box \cdot \Box A_r) \tag{I29}$$

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