EFFECT OF ELASTIC MATERIAL AND GEOMETRIC NONLINEARITIES ON CRACK TIP DEFORMATIONS

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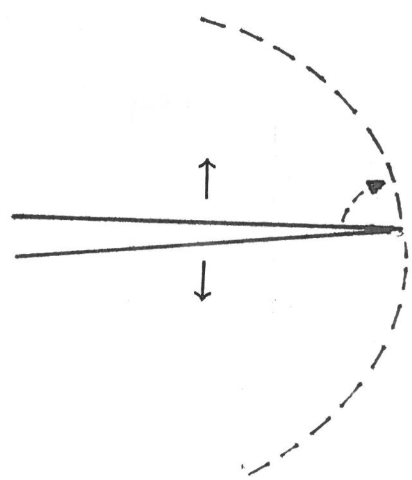
The 2nd Pan American Congress of Applied Mechanics
Valparaiso, Chile
Jan 2-5, 1991

Motivation

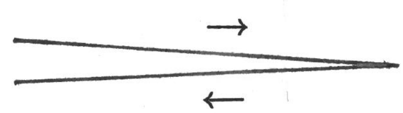
Linear elastic fracture mechanics

Mode I:



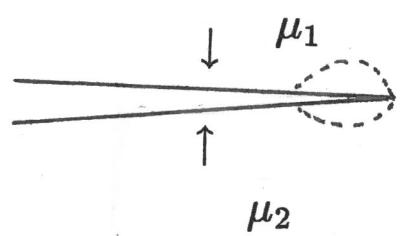
Crack surfaces rotate 90^o, large strains and rotations

Mode II:



Crack surfaces undergo relative sliding, no opening

Interface crack:



Crack surfaces interpenetrate

Some Previous Work

Linear elastic fracture mechanics

- Williams (1957)- asymptotic crack tip stress and displacement fields, homogeneous materials.
- England (1965)- interface crack, stress and displacement fields, oscillatory singularity.
- Rice and Sih(1965)- interface crack, stress and displacement fields, stress intensity factors.

Nonlinear elastic fracture mechanics

- Stephenson (1980)- Mode II crack, incompressible materials, crack opening behavior.
- Knowles (1981)- Mode II crack, compressible material, crack opening predicted, asymptotic behavior.
- Knowles and Sternberg (1983)- interface crack, incompressible materials, showed smooth crack opening behavior, asymptotic solution.
- Ravichandran and Knauss(1989)- interface crack, Neo-Hookean material, FE analysis, full-field solution.
- Lund and Westmann (1990)- Mode I crack, three parameter nonlinear material, FE analysis, full-field solution.

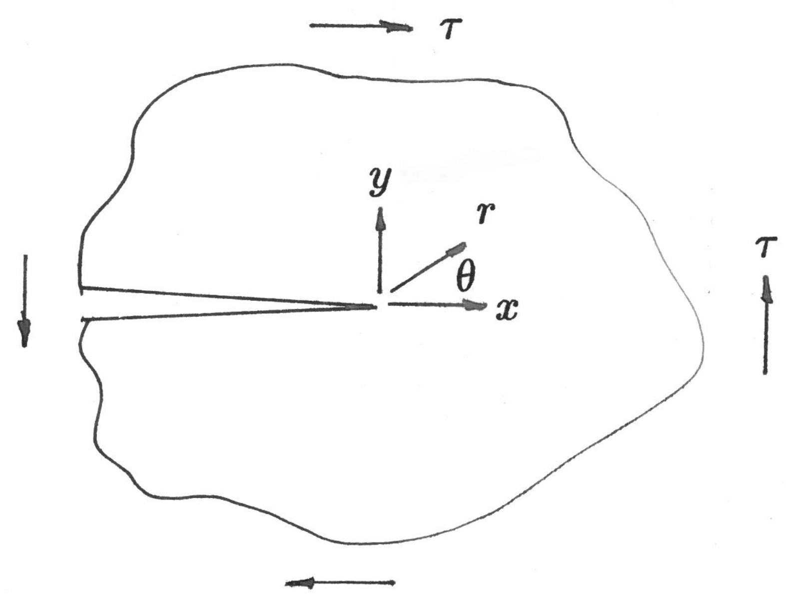
Outline

- Review linear elastic solutions
  - Mode II crack in a homogeneous material
  - Interface crack

- Review nonlinear elasticity theory
  - Classical hyperelastic material (compressible)
  - Neo-Hookean material (incompressible)
  - Blatz-Ko material (compressible)

- Results
  - Mode II crack (linear vs. nonlinear theory)
  - Interface crack (linear vs. nonlinear theory)

Mode II Loading- Linear, homogeneous material



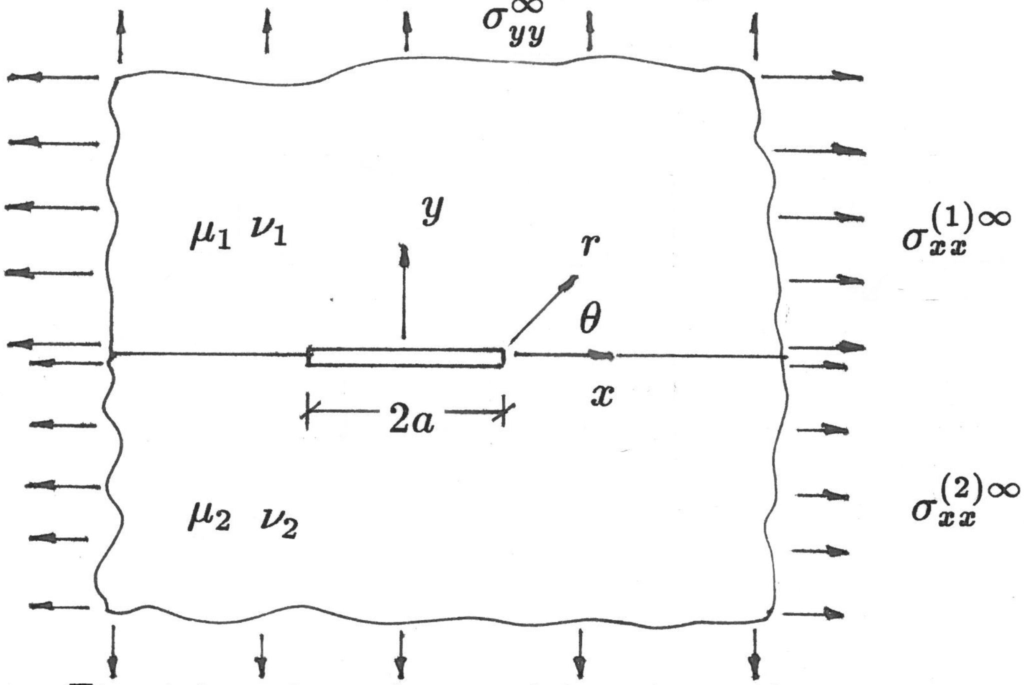
$$u_x = \frac{K_{II}}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left[\frac{\kappa+1}{2}+\cos^2\frac{\theta}{2}\right]+\ldots$$

$$u_y = \frac{K_{II}}{\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left[\frac{1-\kappa}{2}+\sin^2\frac{\theta}{2}\right]+\ldots$$

Conclusion:

- $u_y = 0$ near the crack tip (including higher order terms)

Interface crack, normal loading- Linear, homogeneous materials



$$u_x \sim \sqrt{r}[\phi_x(\theta)\cos(\epsilon \ln r) + \psi_x(\theta)\sin(\epsilon \ln r)] + \dots$$
$$u_y \sim \sqrt{r}[\phi_y(\theta)\cos(\epsilon \ln r) + \psi_y(\theta)\sin(\epsilon \ln r)] + \dots$$

where

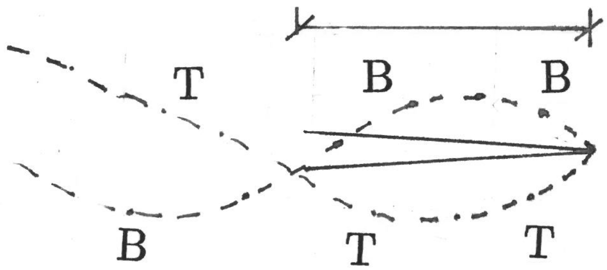
$$\epsilon = \frac{1}{2\pi} \ln\left[\frac{\kappa_1 + \mu_1/\mu_2}{1 + \kappa_2 \mu_1/\mu_2}\right]$$

Bond continuity:

$$(1+\kappa_1)\sigma_{xx}^{(1)\infty} - \mu_1/\mu_2(1+\kappa_2)\sigma_{xx}^{(2)\infty} + [\kappa_1 - 3 - \mu_1/\mu_2(\kappa_2 - 3)]\sigma_{yy}^{\infty} = 0$$

Interpenetration zone (plane stress):

$$r_c = 2a\exp^{-\pi/2\epsilon}$$



<u>Data:</u>

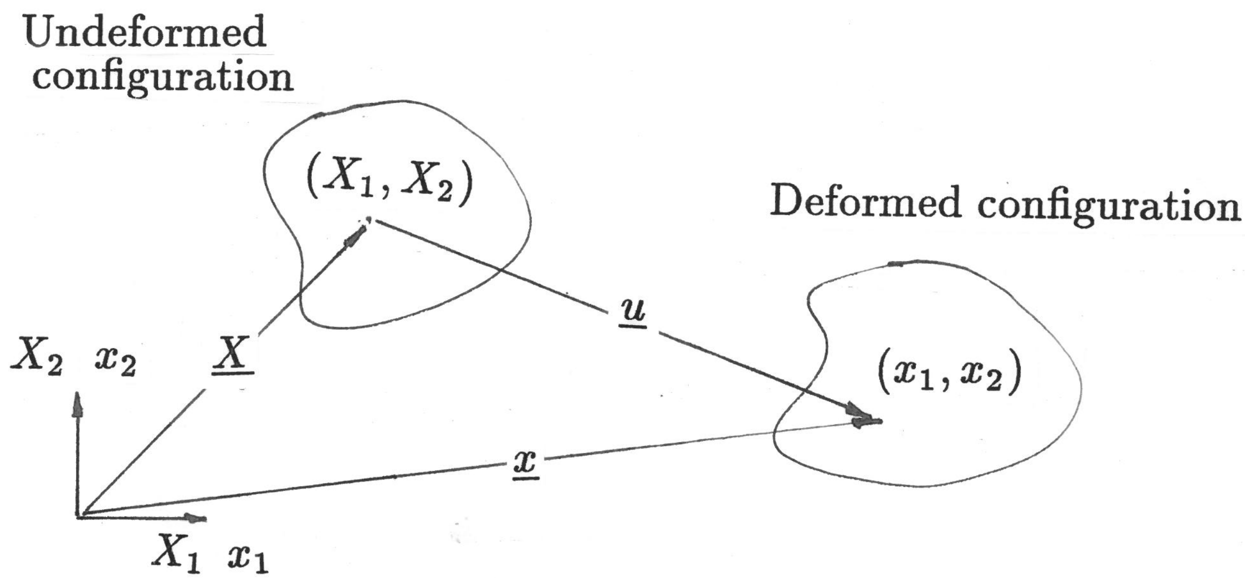
Case *i.)* $\mu_1/\mu_2 = 0.5, \nu_1 = \nu_2 = 0.5$

$$\epsilon = 0.0266 \Rightarrow \frac{r}{2a} = 2.26 \times 10^{-26}$$

Case *ii.)* $\mu_1/\mu_2 = 0.1,\ \nu_1 = 0.1,\ \nu_2 = 0.3$

$$\epsilon = 0.1244 \Rightarrow \frac{r}{2a} = 3.28 \times 10^{-6}$$

Nonlinear Elasticity (2D)



$$x_1 = X_1 + u_1(X_1, X_2)$$
$$x_2 = X_2 + u_2(X_1, X_2)$$

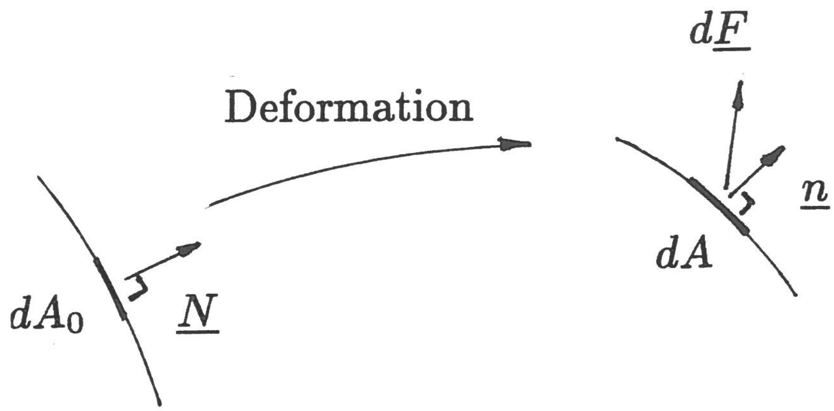
Deformation gradient/ Lagrange strain:

$$F_{iA} \equiv \frac{\partial x_i}{\partial X_A} = \delta_{iA} + \frac{\partial u_i}{\partial X_A}$$

$$det\underline{F} = (F_{11}F_{22} - F_{12}F_{21})\lambda_3$$

$$E_{AB} \equiv \frac{1}{2}(F_{iA}F_{iB} - \delta_{AB}) \quad \text{or} \quad \underline{E} = \frac{1}{2}(\underline{F}^T\underline{F} - \underline{I})$$

Stress tensors:



Cauchy: $\underline{\sigma}$ or σ_{ij}

$$\underline{t} = \frac{d\underline{F}}{dA} = \underline{\sigma n}$$

1st Piola-Kirchhoff: $\underline{P}$ or P_{iA}

$$\underline{t}^0 = \frac{d\underline{F}}{dA_0} = \underline{PN}$$

2nd Piola-Kirchhoff: $\underline{S}$ or S_{AB}

$$\tilde{\underline{t}} = \underline{F}^{-1}\frac{d\underline{F}}{dA_0} = \underline{SN}$$

Transformations:

$$\underline{\sigma} = \frac{1}{(det\underline{F})}\underline{PF^T} \qquad \underline{\sigma} = \frac{1}{(det\underline{F})}\underline{FSF^T}$$

Equilibrium equations:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

where

$$u_i = g_i(\underline{x}) \quad \text{on the boundary } \Gamma_g$$
$$\sigma_{ij} n_j = t_i(\underline{x}) \quad \text{on the boundary } \Gamma_h$$

Weak form:

$$\int_{\Omega} w_i \frac{\partial \sigma_{ij}}{\partial x_j} d\Omega = 0$$

$$\vdots$$

$$\int_{\Omega} \frac{\partial w_i}{\partial x_j} \sigma_{ij} d\Omega = \int_{\Gamma_h} w_i \sigma_{ij} n_j d\Gamma \quad \text{(Updated Lagrange)}$$

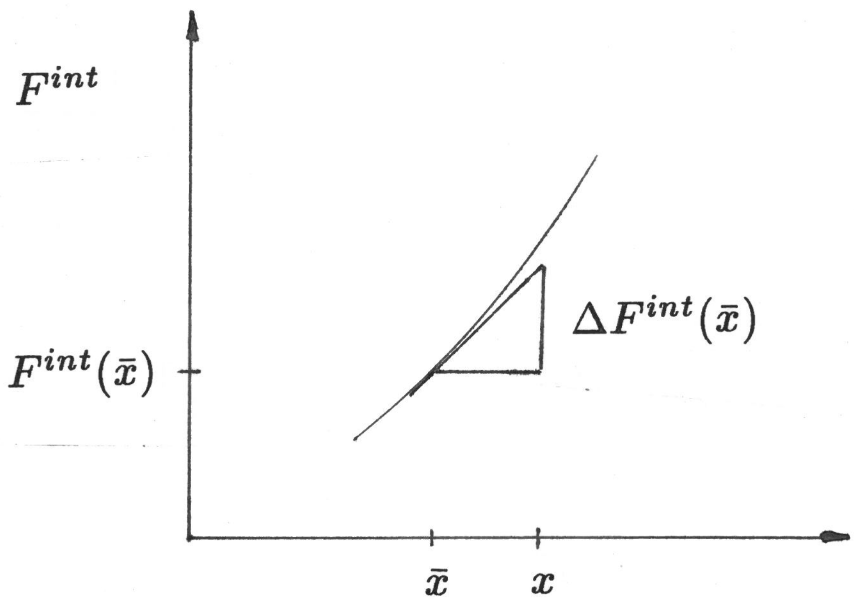
or

$$\int_{\Omega_0} \frac{\partial w_i}{\partial X_A} P_{iA} d\Omega_0 = \int_{\Gamma_h} w_i P_{iA} N_A d\Gamma_0 \quad \text{(Total Lagrange)}$$

$$F^{int}(\underline{x}) = F^{ext}(\underline{x})$$

Finite element equations:

$$F^{int}(\underline{x}) \approx F^{int}(\underline{\bar{x}}) + \Delta F^{int}(\underline{\bar{x}})$$



$$\underline{K}\Delta\underline{d} = \underline{F}^{ext} - \underline{F}^{int}$$

Classical Hyperelastic Material(2D case)

$$S_{AB} = C_{ABCD} E_{CD}$$

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{12} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{12} \end{bmatrix}$$

<u>Blatz-Ko Material</u>(2D case-plane strain)

$$P_{iA} = (2W_{,I} + W_{,J})\delta_{iA} + 2W_{,I}u_{i,A} + W_{,J}\epsilon_{ij}\epsilon_{Ak}u_{j,k}$$

where

$$W_{,I} = \frac{\mu}{2J^2} \qquad W_{,J} = \mu(1 - \frac{I}{J^3})$$

$$I = 2 + 2u_{i,A}\delta_{iA} + u_{i,A}u_{i,A}$$

$$J = det\underline{F} = 1 + u_{1,1} + u_{2,2} + u_{1,1}u_{2,2} - u_{1,2}u_{2,1}$$

$$\begin{bmatrix} P_{11} \\ P_{22} \\ P_{12} \\ P_{21} \end{bmatrix} = \begin{bmatrix} 2W_{,I} & W_{,J} & 0 & 0 \\ W_{,J} & 2W_{,I} & 0 & 0 \\ 0 & 0 & 2W_{,I} & -W_{,J} \\ 0 & 0 & -W_{,J} & 2W_{,I} \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{22} \\ F_{12} \\ F_{21} \end{bmatrix}$$

Neo-Hookean Material(2D case-plane stress)

$$P_{iA} = \mu(F_{iA} - J^{-2}F^{-1}_{Ai})$$

where

$$J = 1 + u_{1,1} + u_{2,2} + u_{1,1}u_{2,2} - u_{1,2}u_{2,1}$$

$$\begin{bmatrix} P_{11} \\ P_{22} \\ P_{12} \\ P_{21} \end{bmatrix} = \begin{bmatrix} \mu & -\mu/J^3 & 0 & 0 \\ -\mu/J^3 & \mu & 0 & 0 \\ 0 & 0 & \mu & \mu/J^3 \\ 0 & 0 & \mu/J^3 & \mu \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{22} \\ F_{12} \\ F_{21} \end{bmatrix}$$

Christensen's Material(2D case-plane strain)

$$P_{iA} = \frac{3}{2}K(J^{2/3} - 1)F^{-1}_{Ai} + \mu(F_{iA} - J^{2/3}F^{-1}_{Ai})$$

where

$$J = 1 + u_{1,1} + u_{2,2} + u_{1,1}u_{2,2} - u_{1,2}u_{2,1}$$

$$\begin{bmatrix} P_{11} \\ P_{22} \\ P_{12} \\ P_{21} \end{bmatrix} = \begin{bmatrix} \mu & Z & 0 & 0 \\ Z & \mu & 0 & 0 \\ 0 & 0 & \mu & -Z \\ 0 & 0 & -Z & \mu \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{22} \\ F_{12} \\ F_{21} \end{bmatrix}$$

where

$$Z = -\frac{3}{2}KJ^{-1} + (\frac{3}{2}K - \mu)J^{-1/3}$$

Tangent stiffness- Blatz-Ko material

$$\Delta F^{int} = \int_{\Omega_0} \begin{bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} \\ w_{2,1} \end{bmatrix}^T \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} \begin{bmatrix} \Delta u_{1,1} \\ \Delta u_{2,2} \\ \Delta u_{1,2} \\ \Delta u_{2,1} \end{bmatrix} d\Omega_0$$

where

$$C_{11} = 2(1+u_{1,1})I_{11} + (1+u_{2,2})J_{11} + 2W_{,I}$$
$$C_{12} = 2(1+u_{1,1})I_{22} + (1+u_{2,2})J_{22} + W_{,J}$$
$$C_{13} = 2(1+u_{1,1})I_{12} + (1+u_{2,2})J_{12}$$
$$C_{14} = 2(1+u_{1,1})I_{21} + (1+u_{2,2})J_{21}$$
$$C_{22} = 2(1+u_{2,2})I_{22} + (1+u_{1,1})J_{22} + 2W_{,I}$$
$$C_{23} = 2(1+u_{2,2})I_{12} + (1+u_{1,1})J_{12}$$
$$C_{24} = 2(1+u_{2,2})I_{21} + (1+u_{1,1})J_{21}$$
$$C_{33} = 2u_{1,2}I_{12} - u_{2,1}J_{12} + 2W_{,I}$$
$$C_{34} = 2u_{1,2}I_{21} - u_{2,1}J_{21} - W_{,J}$$
$$C_{44} = 2u_{2,1}I_{21} - u_{1,2}J_{21} + 2W_{,I}$$

$$I_{11} = -\mu J^{-3}(1 + u_{2,2})$$
$$I_{22} = -\mu J^{-3}(1 + u_{1,1})$$
$$I_{12} = \mu J^{-3} u_{2,1}$$
$$I_{21} = \mu J^{-3} u_{1,2}$$
$$J_{11} = -2\mu J^{-3}(1 + u_{1,1}) + 3\mu I J^{-4}(1 + u_{2,2})$$
$$J_{22} = -2\mu J^{-3}(1 + u_{2,2}) + 3\mu I J^{-4}(1 + u_{1,1})$$
$$J_{12} = -2\mu J^{-3} u_{1,2} - 3\mu I J^{-4} u_{2,1}$$
$$J_{21} = -2\mu J^{-3} u_{2,1} - 3\mu I J^{-4} u_{1,2}$$

Tangent stiffness- Neo Hookean material

$$\Delta F^{int} = \int_{\Omega_0} \begin{bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} \\ w_{2,1} \end{bmatrix}^T \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} \begin{bmatrix} \Delta u_{1,1} \\ \Delta u_{2,2} \\ \Delta u_{1,2} \\ \Delta u_{2,1} \end{bmatrix} d\Omega_0$$

. where

$$\begin{aligned}
C_{11} &= 3\mu J^{-4}(1+u_{2,2})^2 + \mu \\
C_{12} &= 3\mu J^{-4}(1+u_{1,1})(1+u_{2,2}) - \mu J^{-3} \\
C_{13} &= -3\mu J^{-4}u_{2,1}(1+u_{2,2}) \\
C_{14} &= -3\mu J^{-4}u_{1,2}(1+u_{2,2}) \\
C_{22} &= 3\mu J^{-4}(1+u_{1,1})^2 + \mu \\
C_{23} &= -3\mu J^{-4}u_{2,1}(1+u_{1,1}) \\
C_{24} &= -3\mu J^{-4}u_{1,2}(1+u_{1,1}) \\
C_{33} &= 3\mu J^{-4}u_{2,1}u_{2,1} + \mu \\
C_{34} &= 3\mu J^{-4}u_{1,2}u_{2,1} + \mu J^{-3} \\
C_{44} &= 3\mu J^{-4}u_{1,2}u_{1,2} + \mu
\end{aligned}$$

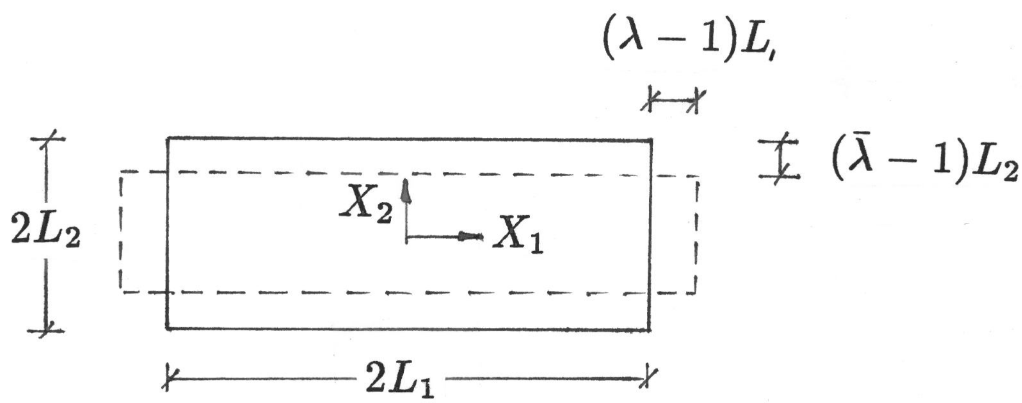
Tangent stiffness- Christensen's material

$$\Delta F^{int} = \int_{\Omega_0} \begin{bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} \\ w_{2,1} \end{bmatrix}^T \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} \begin{bmatrix} \Delta u_{1,1} \\ \Delta u_{2,2} \\ \Delta u_{1,2} \\ \Delta u_{2,1} \end{bmatrix} d\Omega_0$$

where

$$\begin{aligned}
C_{11} &= \mu + Y(1+u_{2,2})^2 \\
C_{12} &= Z + Y(1+u_{1,1})(1+u_{2,2}) \\
C_{13} &= -Yu_{2,1}(1+u_{2,2}) \\
C_{14} &= -Yu_{1,2}(1+u_{2,2}) \\
C_{22} &= \mu + Y(1+u_{1,1})^2 \\
C_{23} &= -Yu_{2,1}(1+u_{1,1}) \\
C_{24} &= -Yu_{1,2}(1+u_{1,1}) \\
C_{33} &= \mu + Yu_{2,1}^2 \\
C_{34} &= -Z + Yu_{1,2}u_{2,1} \\
C_{44} &= \mu + Yu_{1,2}^2 \\
Y &= \frac{3}{2}KJ^{-2} + (-\frac{K}{2} + \frac{\mu}{3})J^{-4/3}
\end{aligned}$$

Uniaxial response

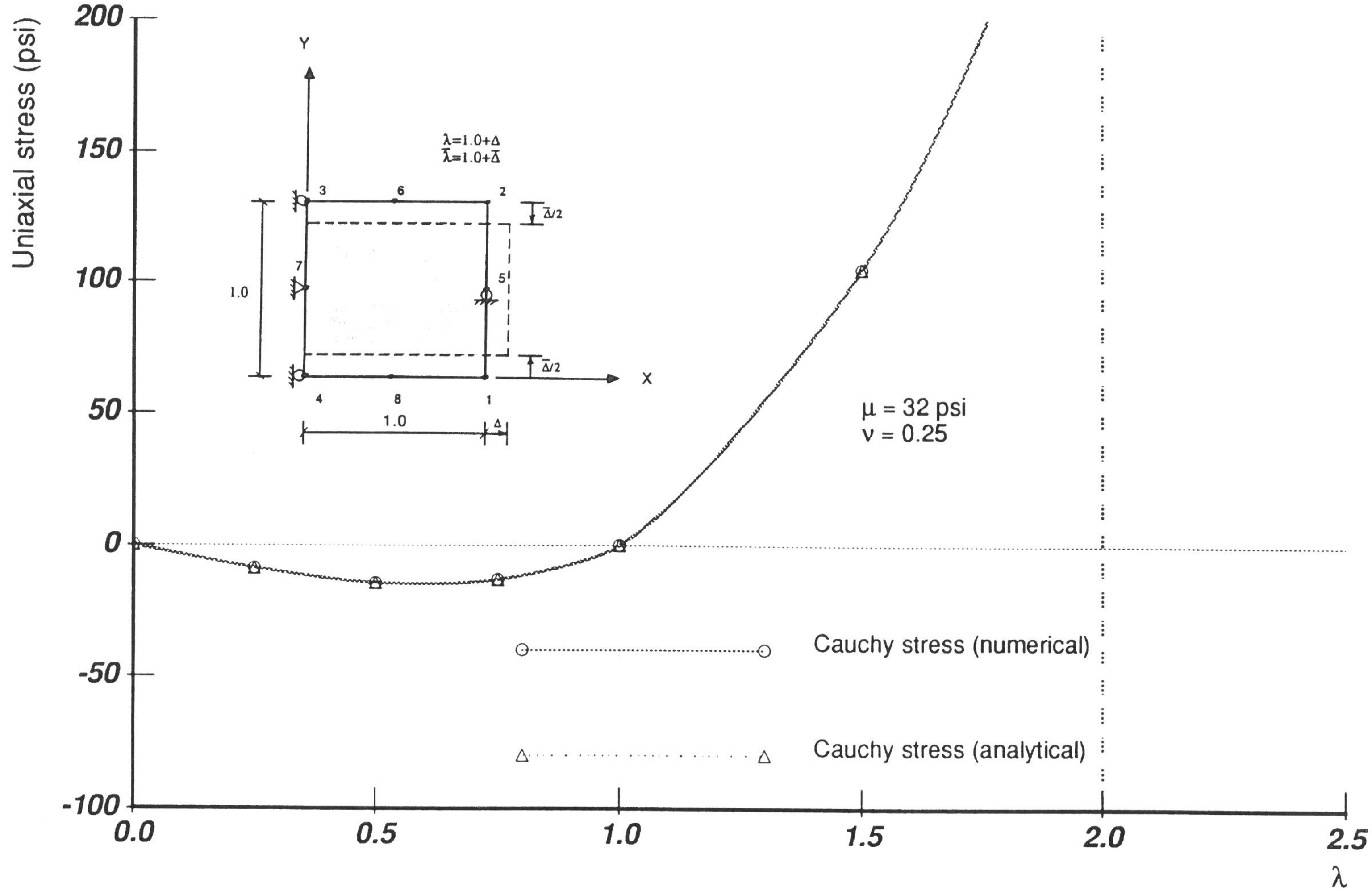


Case 1: Classical hyperelastic, Blatz-Ko, or Chistensen (plane strain)

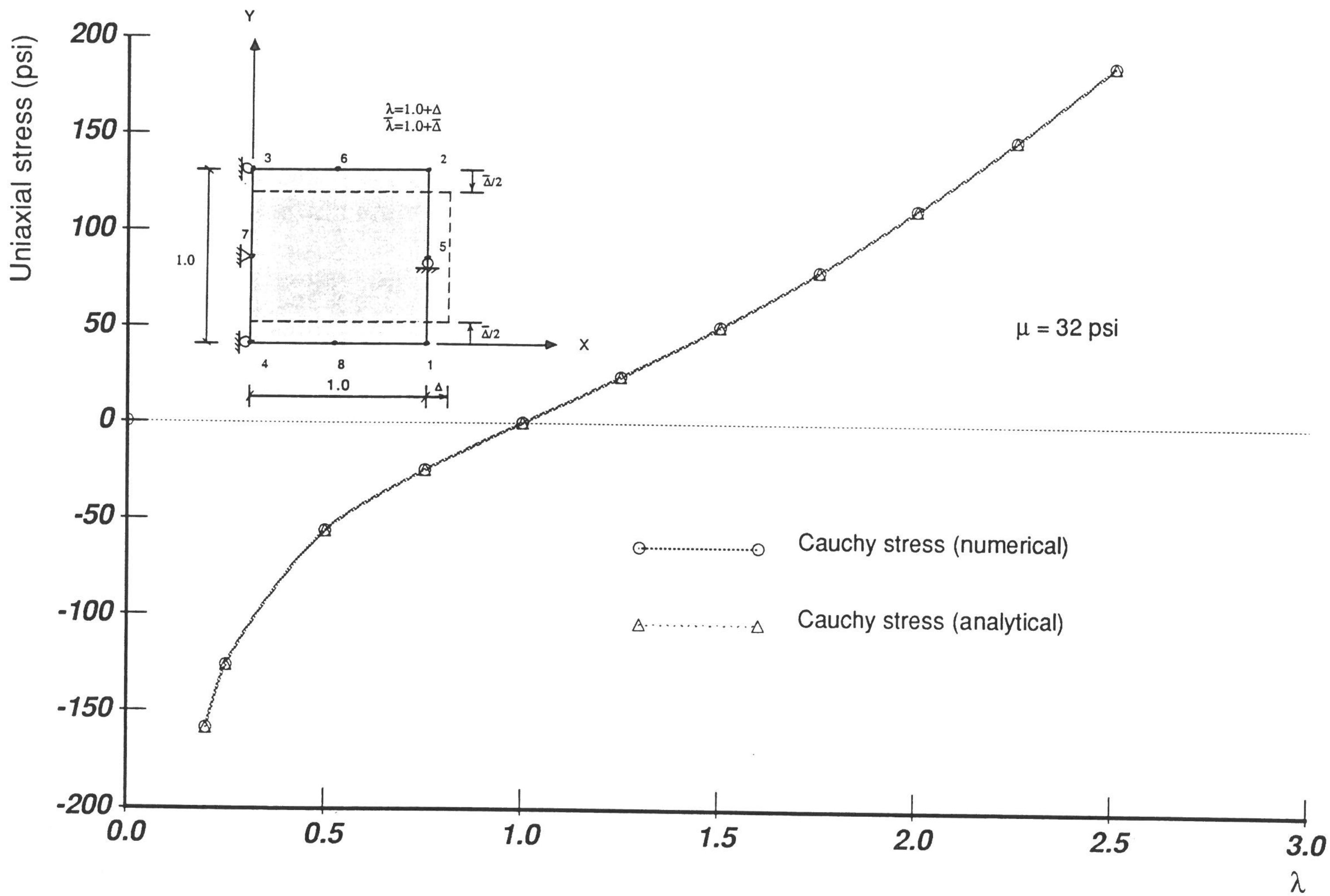
$$
\begin{aligned}
u_1 &= (\lambda - 1)X_1 \\
u_2 &= (\bar{\lambda} - 1)X_2 \\
u_3 &= 0
\end{aligned}
$$

Case 2: Neo-Hookean (plane stress)

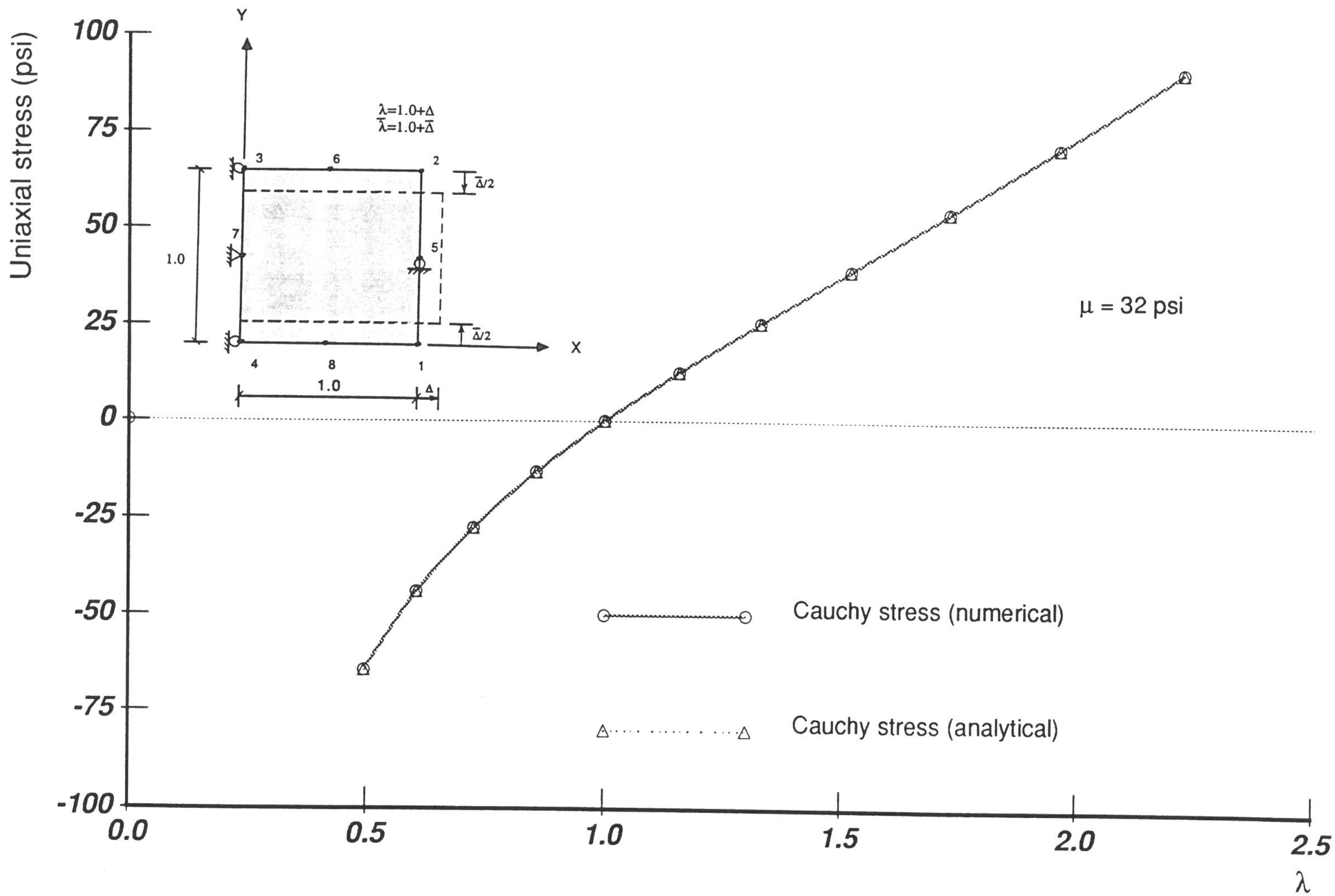
$$
\begin{aligned}
u_1 &= (\lambda - 1)X_1 \\
u_2 &= (1/\sqrt{\lambda} - 1)X_2 \\
u_3 &= (1/\sqrt{\lambda} - 1)X_3
\end{aligned}
$$



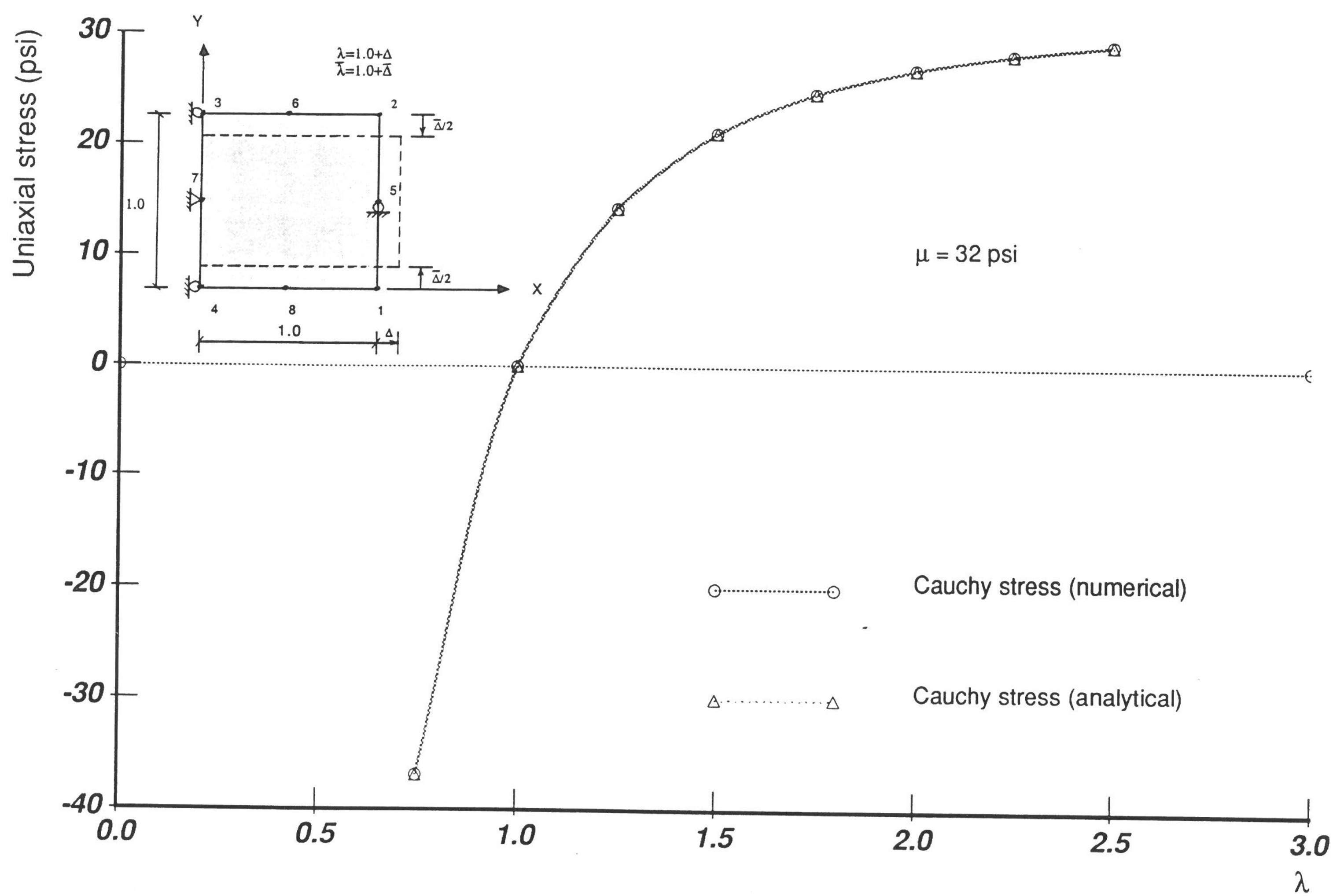
Response of a classical hyperelastic material for a homogeneous sheet (plane strain)



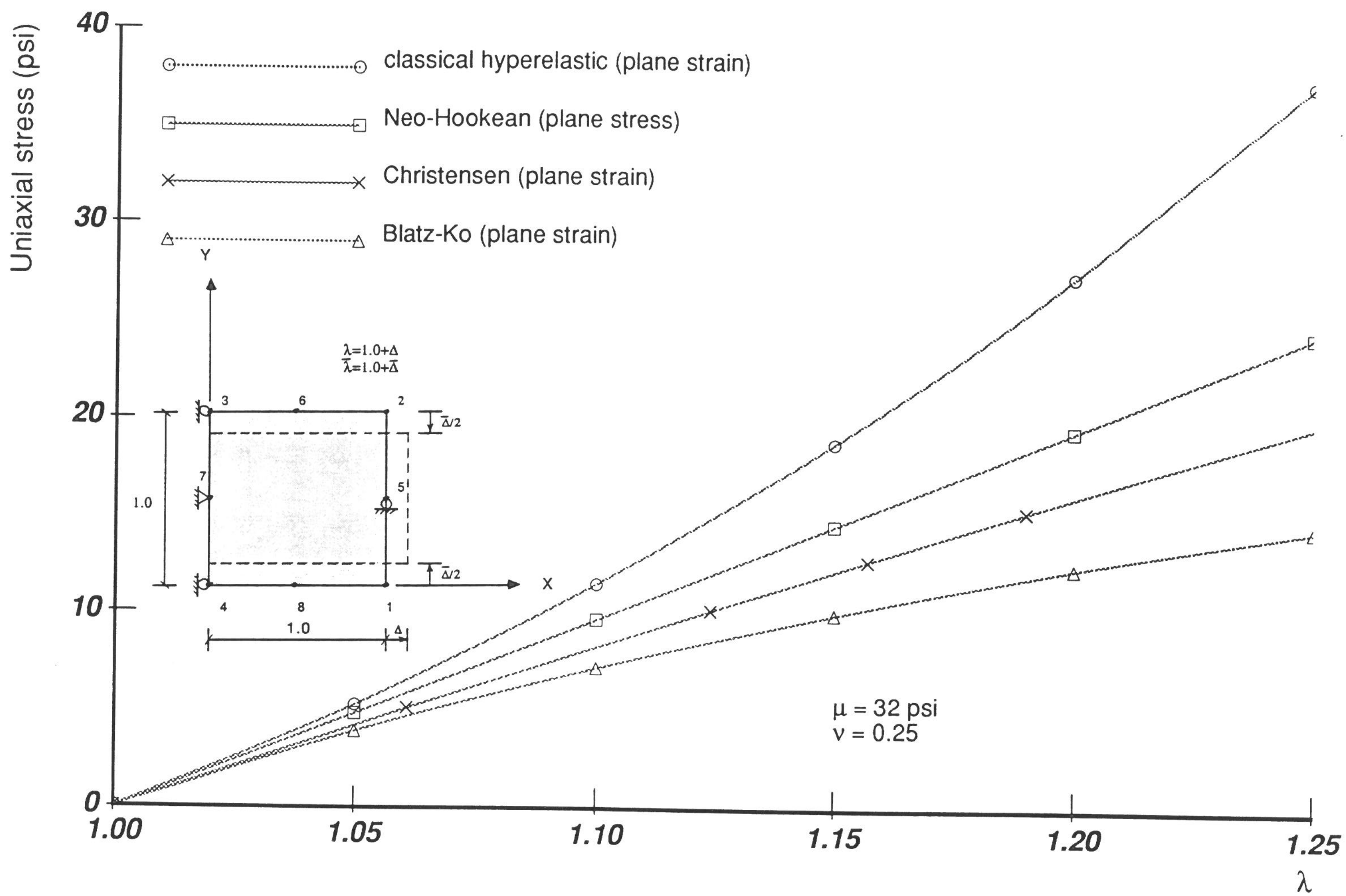
Response of a Neo-Hookean material for a homogeneous sheet (plane stress)



Response of a Christensen material for a homogeneous sheet (plane strain)

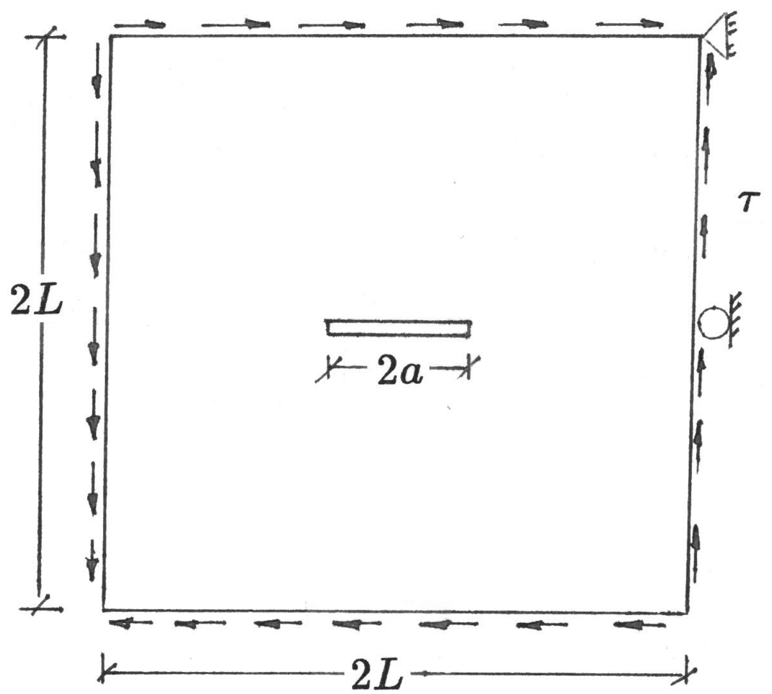


Response of a Blatz-Ko material for a homogeneous sheet (plane strain)



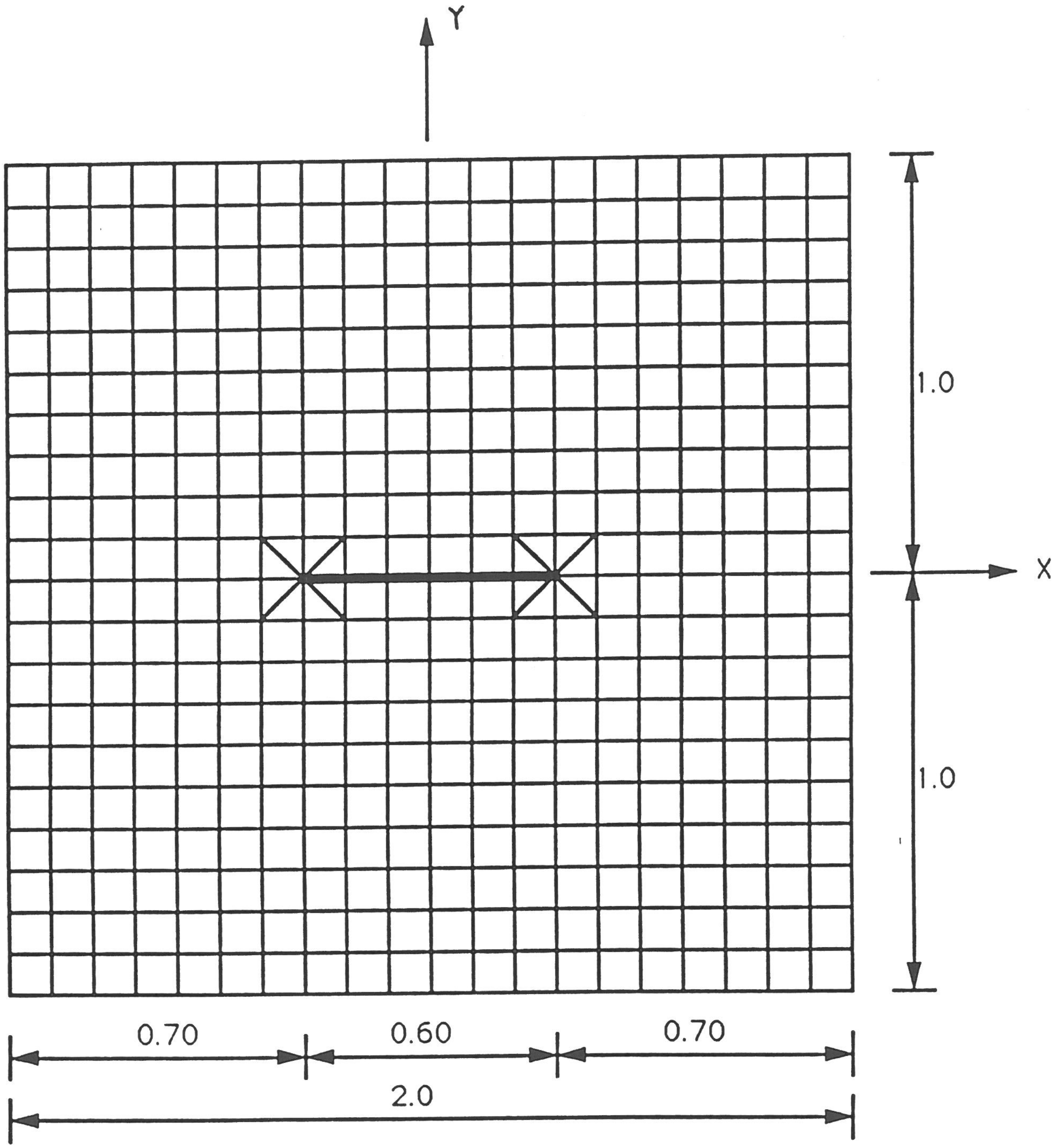
Comparison between response of classical hyperelastic, Neo-Hookean, Christensen and Blatz-Ko materials

Mode II- homogeneous material

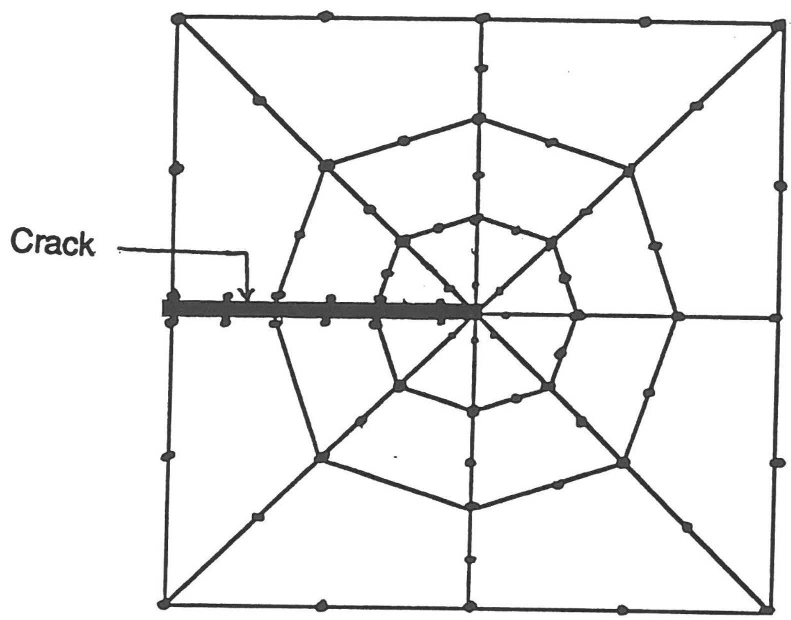


Data:

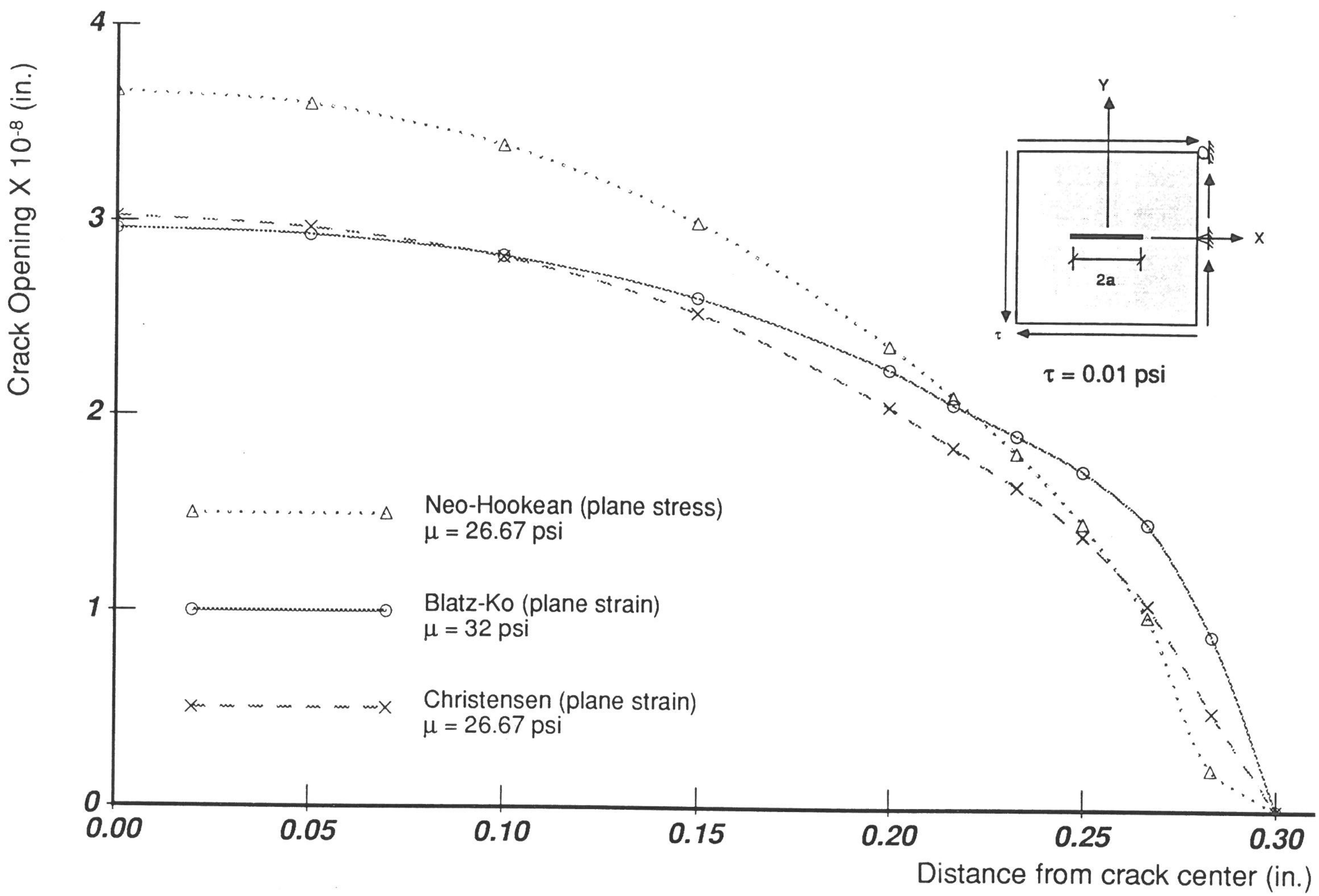
$$2L = 2.0$$
$$2a = 0.6$$
$$E = 80$$
$$\nu = 0.25$$
$$\mu = 32$$
$$\tau = 0.01$$



Global finite element mesh

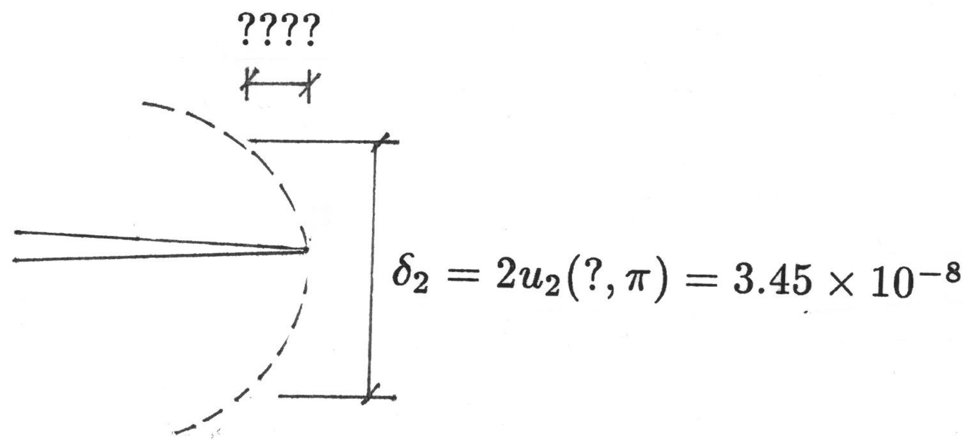


Modelling the crack tip zone



Non-Linear Mode II Problem

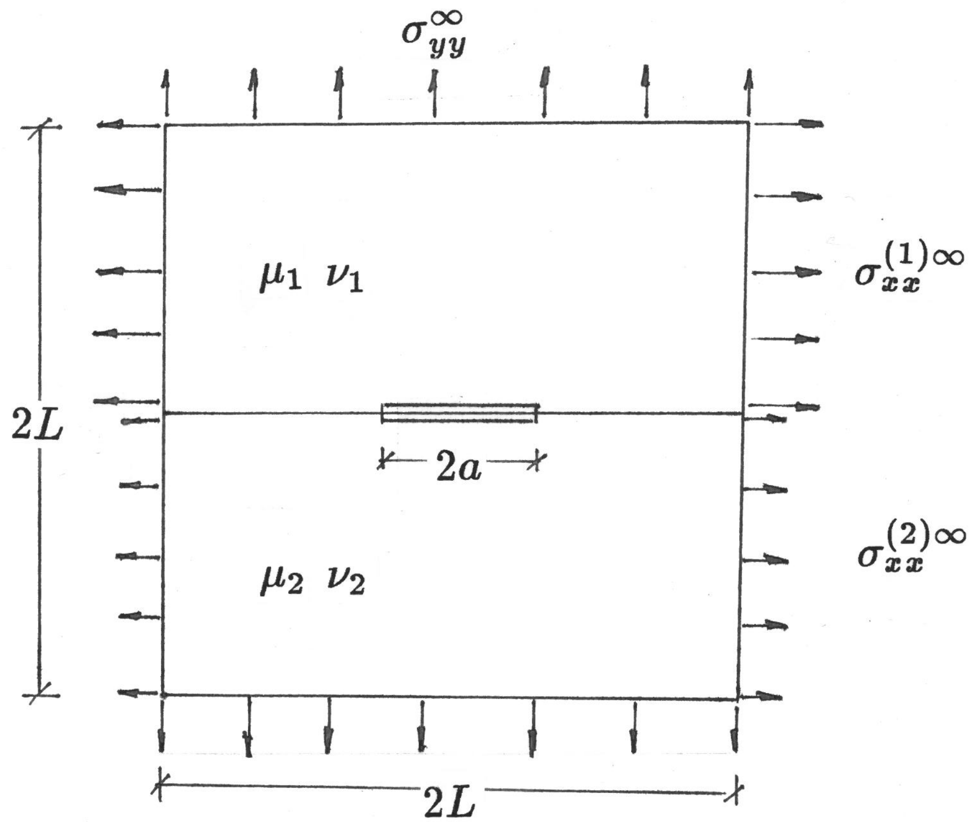
Knowles asymptotic solution



$$u_2(r,\theta) = \underbrace{r^{1/2} U_2(\theta)}_{\text{LEFM}} + \underbrace{r^{m_2} V_2(\theta)}_{\text{NLEFM}} + \ldots$$

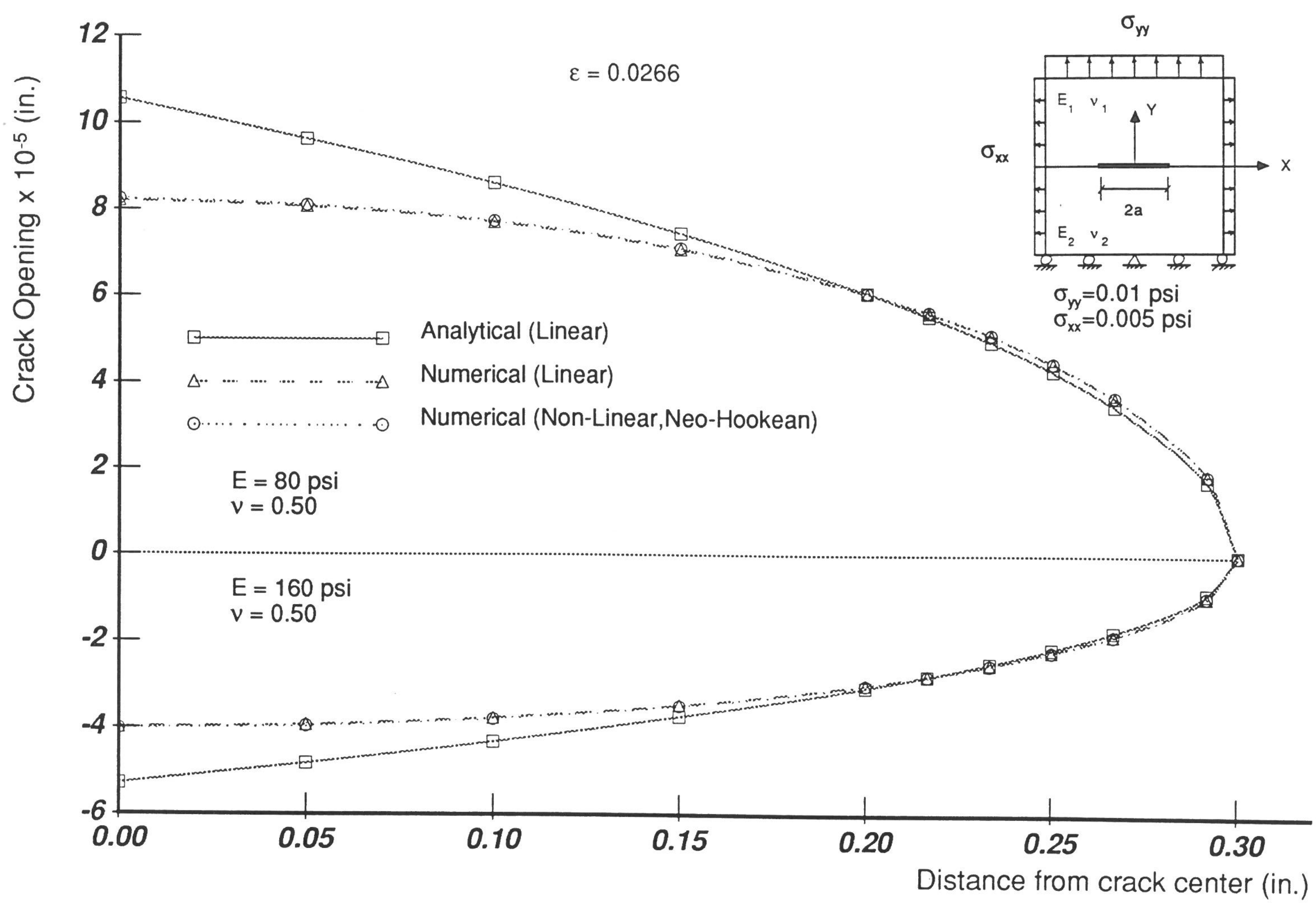
- Numerical calculations show $m_2 = 0.4$, where Knowles *assumed* $m_2 = 0$.

Interface Crack Problem

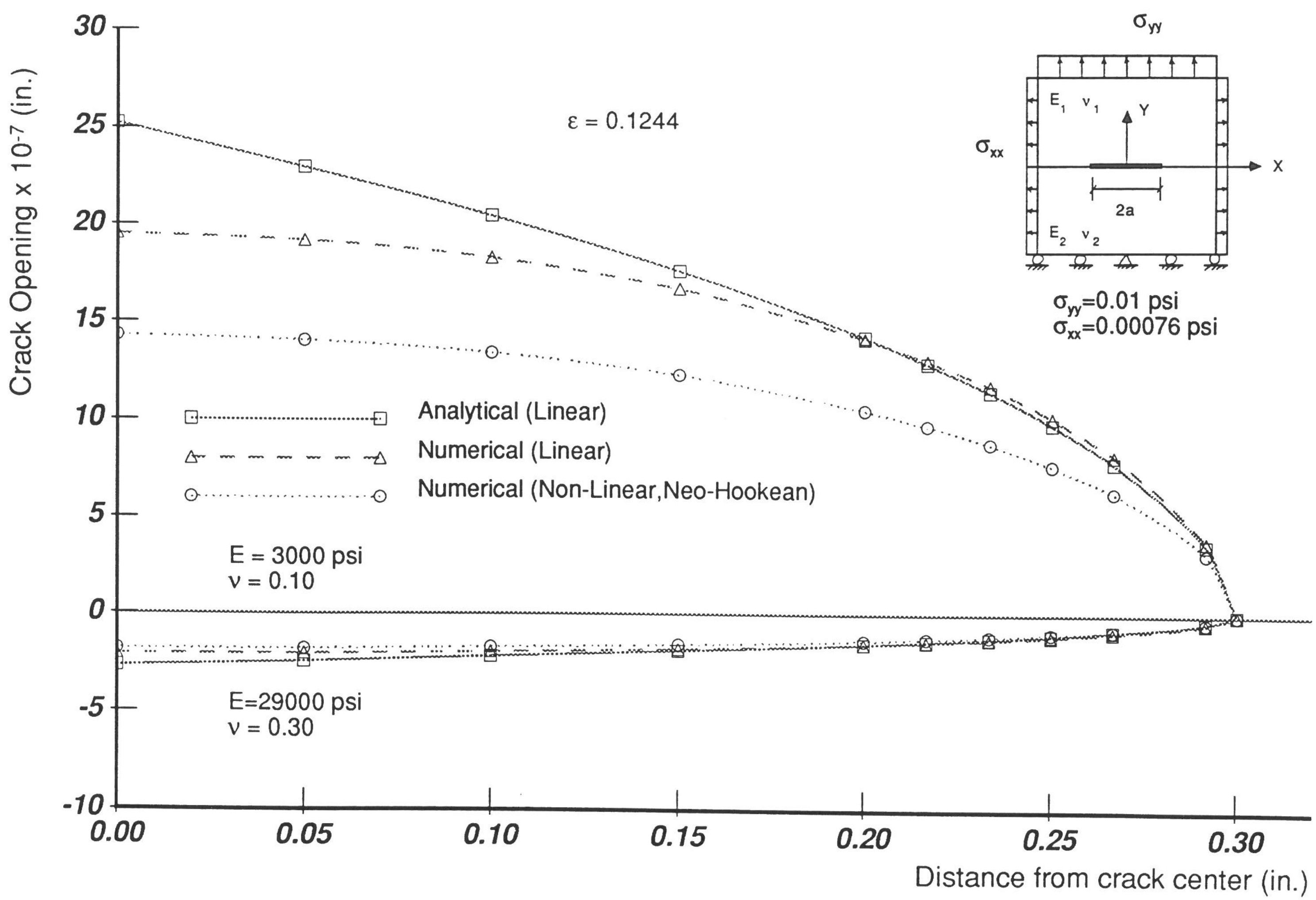


Data:

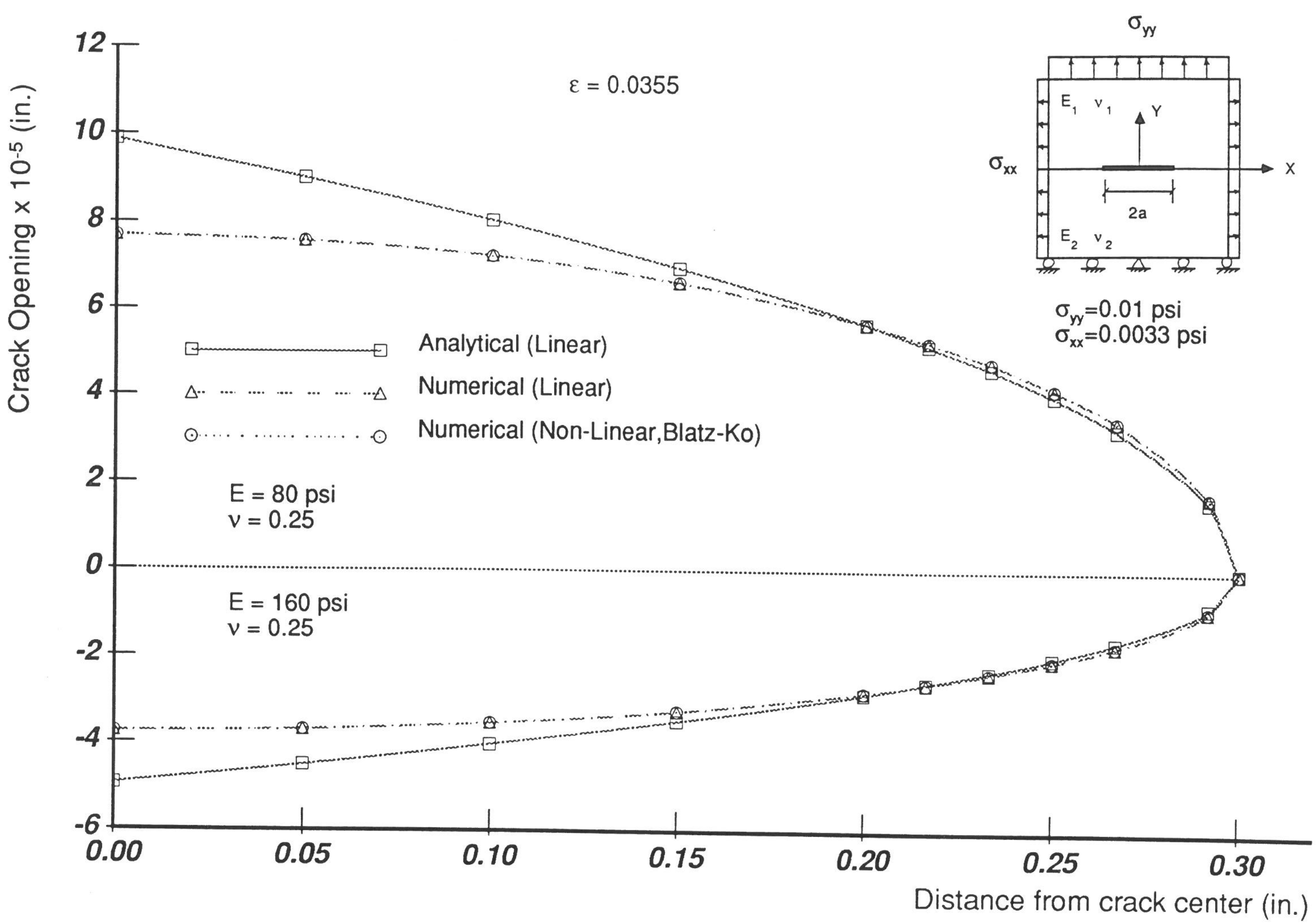
$$2L = 2.0$$
$$2a = 0.6$$
$$E_1 = 80,\ \nu_1 = 0.5$$
$$E_2 = 160\ \nu_2 = 0.5$$
$$E_1 = 3000,\ \nu_1 = 0.1$$
$$E_2 = 29000,\ \nu_2 = 0.3$$
$$\sigma_{xx}^{(1)\infty} = \sigma_{xx}^{(2)\infty} = 7.6 \times 10^{-4}$$
$$\sigma_{yy}^{\infty} = 0.01$$



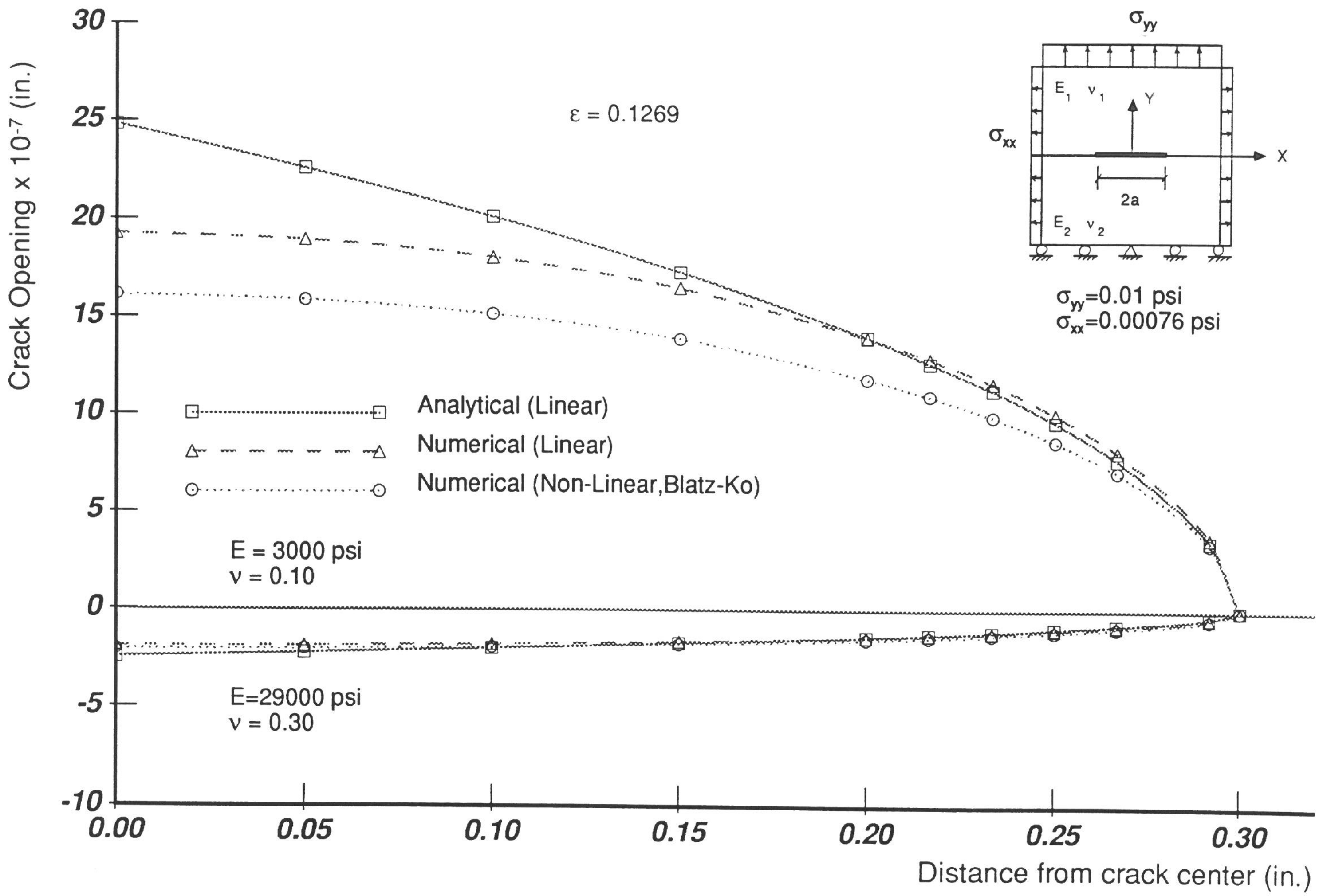
Bimaterial interface crack $\varepsilon = 0.0266$ (plane stress)



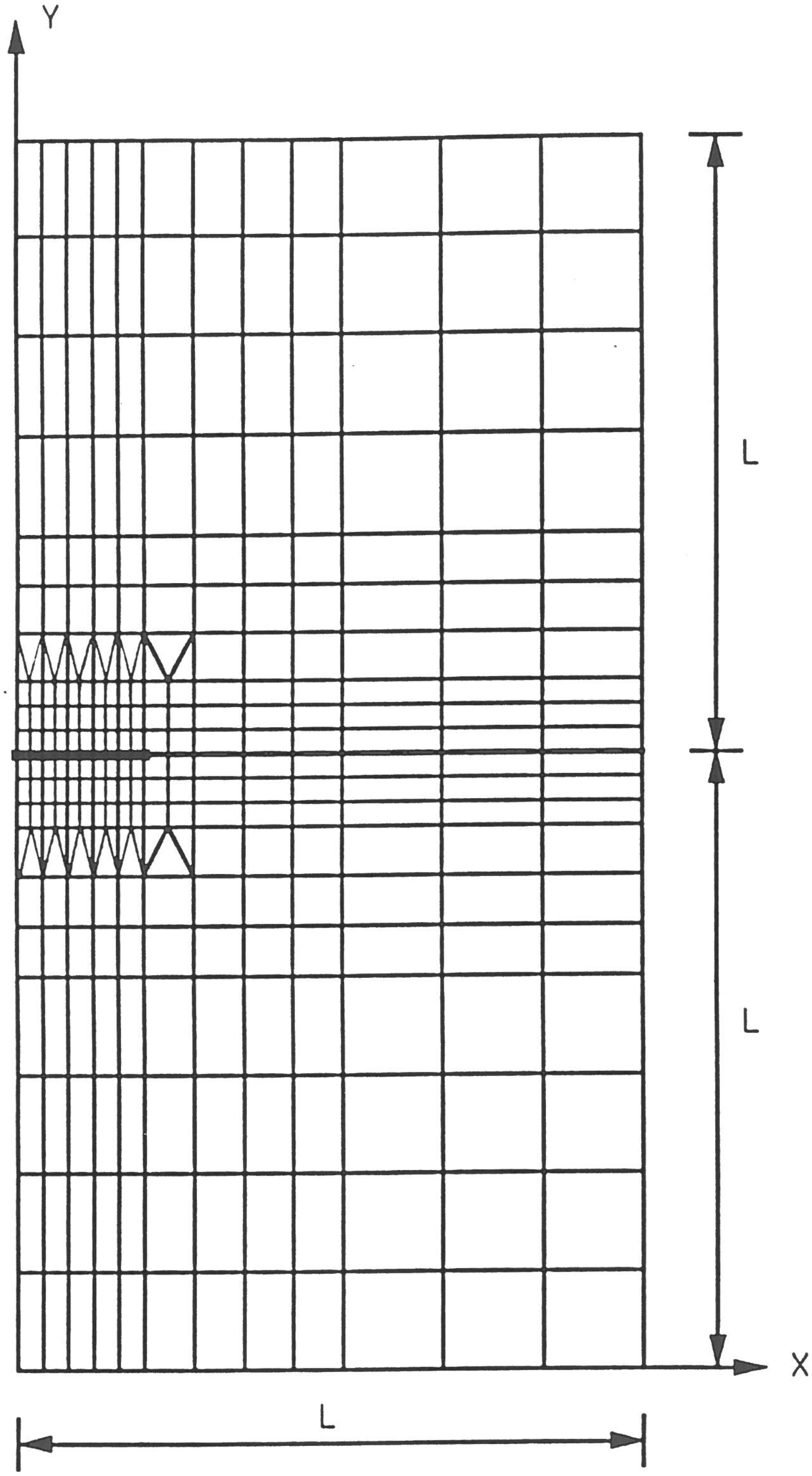
Bimaterial interface crack ε=0.1244 (plane stress)



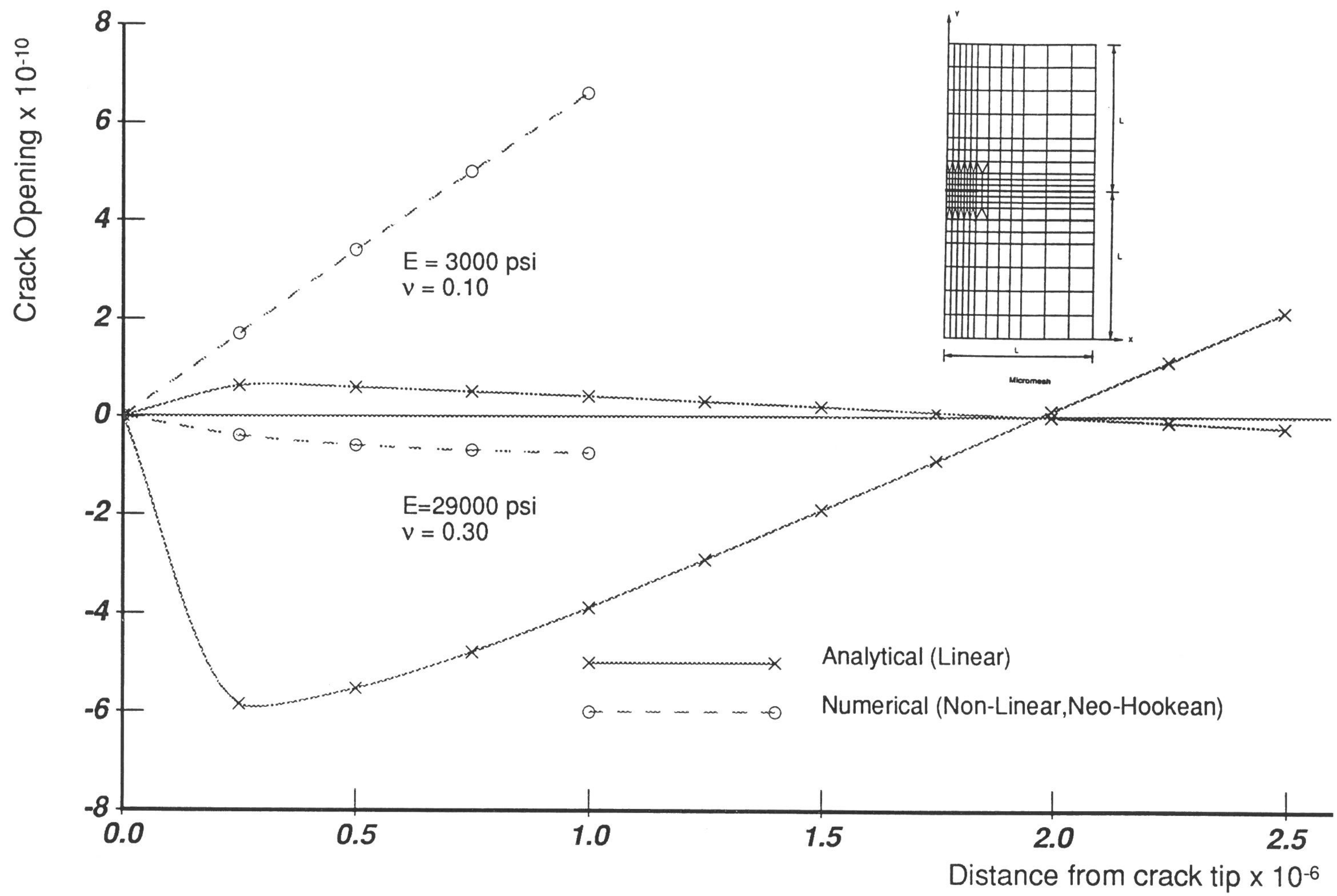
Bimaterial interface crack ε = 0.0355 (plane strain)



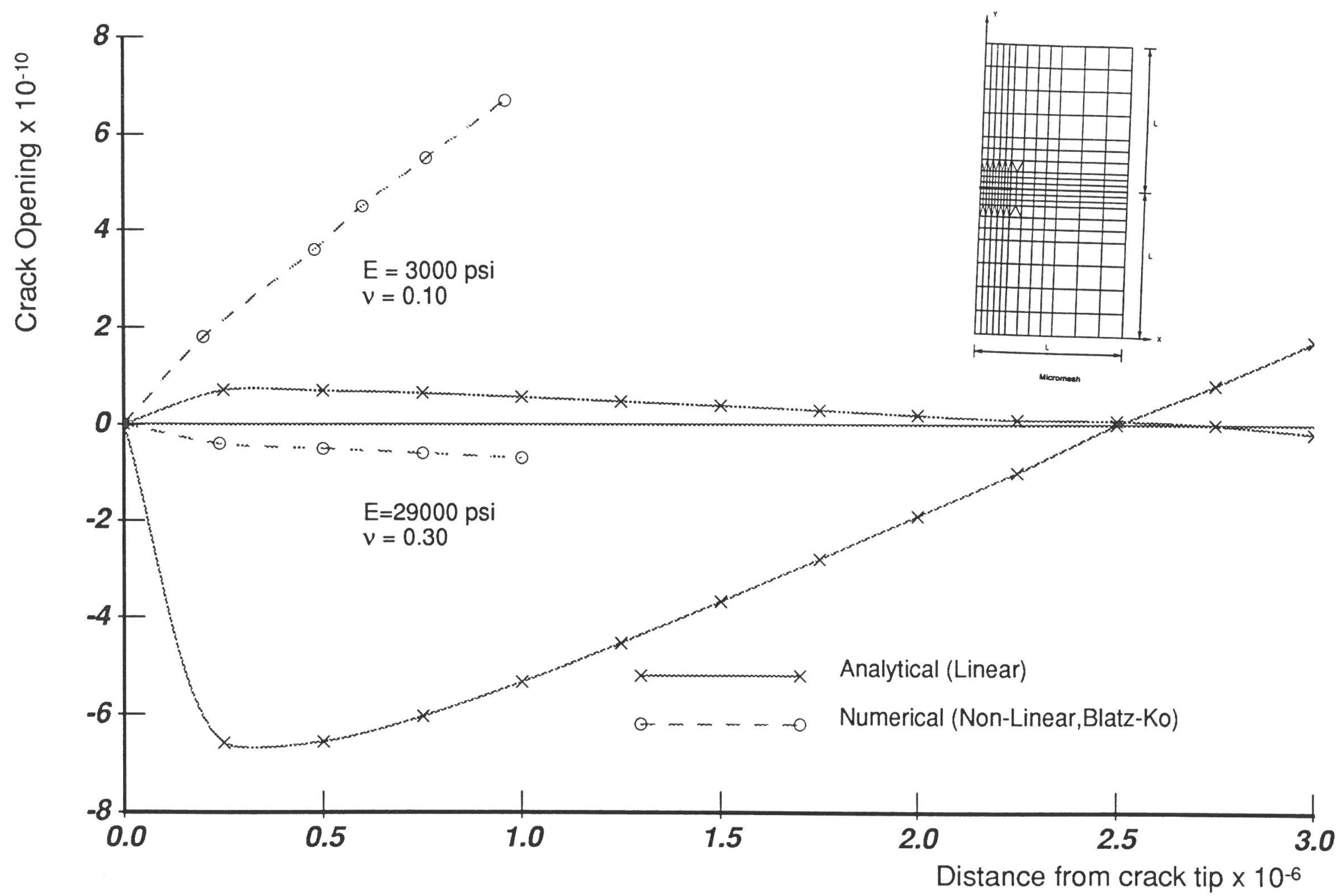
Bimaterial interface crack ε=0.1269 (plane strain)



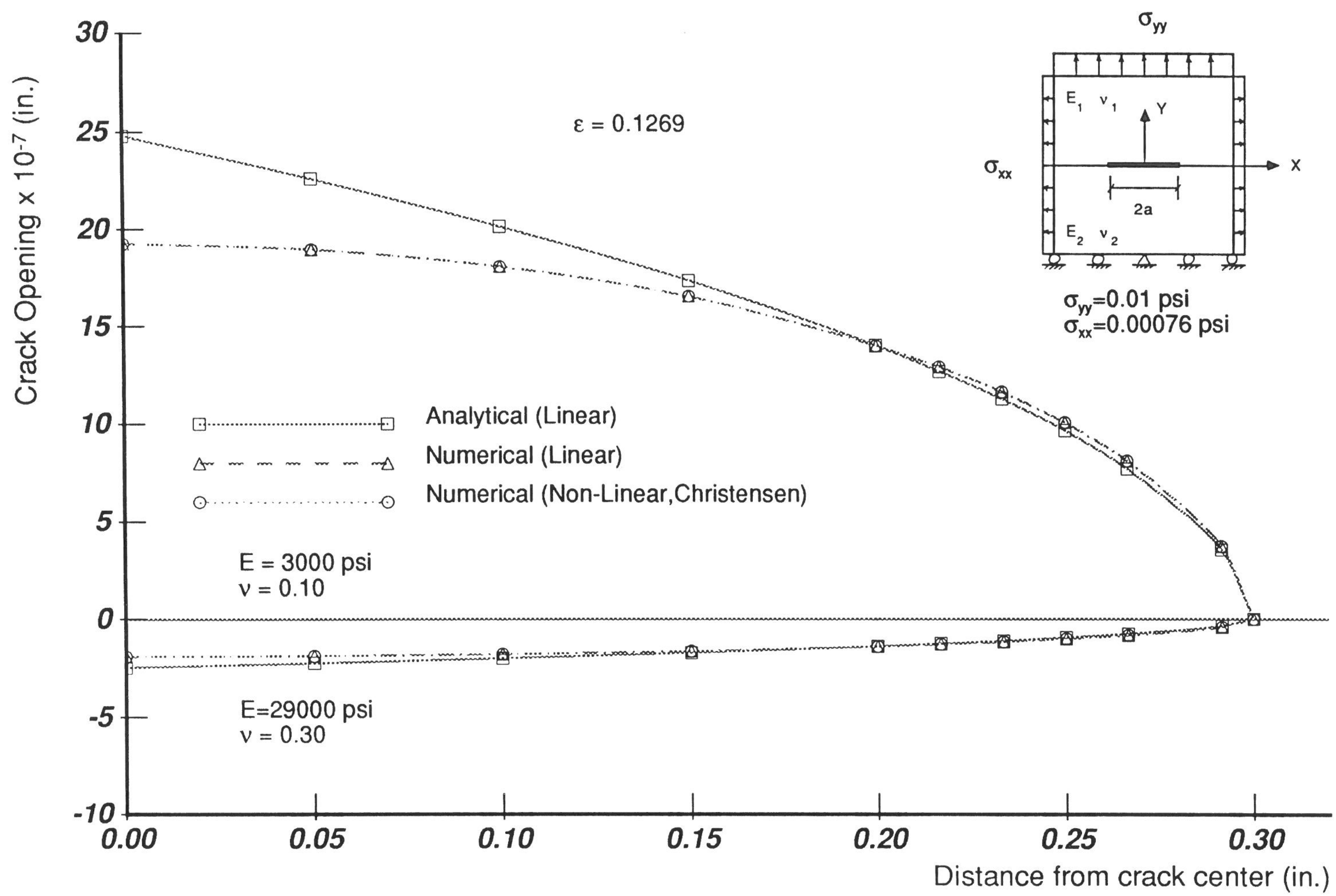
Micromesh



Bimaterial interface crack $\varepsilon = 0.1244$ (plane stress)



Bimaterial interface crack $\varepsilon = 0.1269$ (plane strain)



Bimaterial interface crack ε=0.1269 (plane strain)