Reprinted from June 1987, Vol. 109, Journal of Applied Mechanics

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Energy Release Rates and Related Balance Laws in Linear Elastic Defect Mechanics

A simple and direct derivation of certain balance (or conservation) laws for linear dynamic elasticity is presented including nonhomogeneities, thermal effects, anisotropy, and body forces. Additionally, the connection between the balance laws and energy release rates for defect motions is established.

1 Introduction

Some time ago Knowles and Sternberg (1972) devoted an article to a systematic derivation via Noether's (1971) theorem of certain conservation (or balance) laws valid for linearized and finite elastostatics. This work was motivated primarily by the interest generated by Rice's (1968) J integral, and a quest for its theoretical underpinning. Knowles and Sternberg (1972) deduced that J was actually a vector component of a more general conservation law (J_k), and found two additional laws which were subsequently associated with the L_k and M integrals of defect mechanics by Budiansky and Rice (1973). Fletcher (1976) extended the invariant integrals presented by Knowles and Sternberg (1972) to linear elastodynamics, again relying on manipulations dictated by Noether's (1971) theorem.

It is the purpose of the present paper to extend the results of these earlier works to account for material nonhomogeneity, temperature gradients, anisotropy, and body force. Additionally, it will be shown that balance laws of the type described above can be deduced simply by subjecting the Lagrangian density to familiar operations from vector calculus such as gradient, curl, and divergence. A. G. Herrmann (1981, 1982a,b) introduced the notion of obtaining balance laws in this manner, but did not carry out the steps necessary to include the effects mentioned above. While derivations based on Noether's theorem are admittedly more satisfying from a theoretical standpoint, the attendant mathematical apparatus tends to obscure the relative simplicity of the end results. Therefore, alternative means of deriving balance laws for linear elasticity seems to be a credible objective.

The significance of balance laws would be quite restricted, were it not possible to relate them to energy release rates. These relations are derived here for the special case of a two-dimensional crack and culminate in the conclusion that total energy densities (Hamiltonian) lead directly to physically meaningful energy release rates, rather than Lagrangian densities.

2 Balance Laws

Consider a linear elastic solid occupying a closed, bounded, regular region Ω in three-dimensional space. The material within this region is acted upon by thermal, mechanical, and far-field (e.g., body force) loadings. The measure numbers of the displacement vector $\mathbf{u}$, at the point with position vector $\mathbf{x}$ are given by $u_i = u_i(x_j, t)$.[2] The components of the infinitesimal strain tensor are

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

In the presence of a temperature gradient, the strain tensor is decomposed into two parts: one associated with elastic strains (ϵ_{ij}^E) and another accounting for the free thermal expansion of the medium. Thus,

$$\epsilon_{ij} = \epsilon_{ij}^E + \alpha_{ij}(\mathbf{x})\Delta\Theta(\mathbf{x}) \tag{2}$$

where α_{ij} is a tensor storing the coefficients of thermal expansion, while $\Delta\Theta$ is the temperature change. Both of these quantities are assumed independent of time. According to Neumann's (1841) hypothesis, the stress is related to the elastic strain in the usual way, viz

$$\sigma_{ij} = c_{ijkl}(\mathbf{x})\epsilon_{ij}^E \tag{3}$$

where σ_{ij} are the components of the stress tensor and c_{ijkl} denotes components of the elasticity tensor. Attention is restricted to uncoupled thermoelasticity, consequently material properties are not dependent on the temperature change. Note that the material is nonhomogeneous in general, since the elasticity tensor and coefficients of thermal expansion depend on position. Furthermore, the material is not

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Contributed by the Applied Mechanics Division for presentation at the 1987 Applied Mechanics, Biomechanics, and Fluids Engineering Conference, Cincinnati, OH, June 14–17, 1987, of the American Society of Mechanical Engineers.
Discussion of this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until two months after final publication of the paper itself in the JOURNAL OF APPLIED MECHANICS. Manuscript received by ASME Applied Mechanics Division, June 10, 1986; final revision, November 14, 1986.
Paper No. 87-APM-33.

[2] The *summation convention* is used repeatedly in this paper. Summation over repeated subindices is implied. The range of the subindices is 1,2,3.

necessarily elastically or thermally isotropic. The equilibrium equations are

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i \tag{4}$$

where f_i are measure numbers of the body force density vector, and ρ the mass density. The superposed dots indicate time derivatives. The Lagrangian density L may be viewed as a function of the particle velocity, total strain tensor, and coordinates. Accordingly, it is defined by

$$L = L(\dot{\mathbf{u}},\boldsymbol{\epsilon},\mathbf{x}) \equiv T - W$$

$$= \frac{1}{2}\rho(\mathbf{x})\dot{u}_p\dot{u}_p$$

$$-\frac{1}{2}c_{prst}(\mathbf{x})(\epsilon_{pr} - \alpha_{pr}(\mathbf{x})\Delta\Theta(\mathbf{x}))\epsilon_{st} - \alpha_{st}(\mathbf{x})\Delta\Theta(\mathbf{x})) \tag{5}$$

where T and W represent the kinetic and elastic strain energy densities, respectively. Given the particular form of equation (5), the usual variation of the Lagrangian will lead to the equations of motion, provided terms accounting for the work of the body forces and surface tractions are included.

The desired results are obtained next by simple vector and tensor calculus operations involving the Lagrangian.

Gradient. The first balance law is obtained by considering the gradient of the Lagrangian, i.e.,

$$\nabla L = L_{,k} = \frac{\partial L}{\partial \dot{u}_i}\frac{\partial \dot{u}_i}{\partial x_k} + \frac{\partial L}{\partial \epsilon_{ij}}\frac{\partial \epsilon_{ij}}{\partial x_k} + \left(\frac{\partial L}{\partial x_k}\right)_{\text{expl}}$$

$$= \frac{\partial L}{\partial \dot{u}_i}\frac{\partial \dot{u}_i}{\partial x_k} + \frac{\partial L}{\partial \epsilon^E_{ij}}\frac{\partial \epsilon_{ij}}{\partial x_k} + \left(\frac{\partial L}{\partial x_k}\right)_{\text{expl}}$$

$$= \rho\dot{u}_i\dot{u}_{i,k} - \sigma_{ij}u_{i,jk} + (L_{,k})_{\text{expl}} \tag{6}$$

The "explicit" derivation of the Lagrangian is calculated according to

$$\left(\frac{\partial L}{\partial x_k}\right)_{\text{expl}} = \frac{\partial}{\partial x_k}L(\dot{u}_i,\epsilon_{ij},x_i)\Big|_{\dot{u}_i=\text{const},\epsilon_{ij}=\text{const},x_i=\text{const for } i\neq k} \tag{7}$$

Specifically,

$$(L_{,k})_{\text{expl}} = \frac{1}{2}\rho_{,k}\dot{u}_p\dot{u}_p - \frac{1}{2}c_{prst,k}\epsilon^E_{pr}\epsilon^E_{st} + \sigma_{pr}\alpha_{pr,k}\Delta\Theta + \sigma_{pr}\alpha_{pr}\Delta\Theta_{,k} \tag{8}$$

When the divergence of the stress tensor is introduced in equation (6), it follows that

$$L_{,j}\delta_{jk} + (\sigma_{ij}u_{i,k})_{,j} - \sigma_{ij,j}u_{i,k} - \rho\dot{u}_i\dot{u}_{i,k} = (L_{,k})_{\text{expl}} \tag{9}$$

The equilibrium equations are then substituted producing

$$(L\delta_{jk} + \sigma_{ij}u_{i,k})_{,j} - (\rho\ddot{u}_i - f_i)u_{i,k} - \rho\dot{u}_i\dot{u}_{i,k} = (L_{,k})_{\text{expl}} \tag{10}$$

Finally, it follows that

$$(L\delta_{jk} + \sigma_{ij}u_{i,k})_{,j} = \frac{\partial}{\partial t}(\rho\dot{u}_i u_{i,k}) + (L_{,k})_{\text{expl}} - f_i u_{i,k} \tag{11}$$

This differential form represents a balance law valid for a linear, nonhomogeneous, anisotropic solid which is subjected to inertial, thermal, and body force loadings. If nonhomogeneity, thermal gradients, and body force loadings are absent, and the material is homogeneous, the resulting expression is in accord with equation (3.4) of the paper by Fletcher (1976). An integral form of equation (11) may be obtained upon application of the divergence theorem. If Ω is a regular bounded region enclosed by a surface Γ whose unit outward normal vector is $\mathbf{n}$, it follows that

$$\int_\Gamma(-Ln_k - \sigma_{ij}n_j u_{i,k}k)d\Gamma - \int_\Omega\left[-\frac{\partial}{\partial t}(\rho\dot{u}_i u_{i,k}) - (L_{,k})_{\text{expl}} + f_i u_{i,k}\right]d\Omega = 0 \tag{12}$$

Curl. The second balance law is obtained by considering the curl of the "Lagrangian moment." That is,

$$\nabla\times(L\mathbf{x}) = e_{kij}(Lx_j)_{,i}$$

$$= e_{kij}\left[\frac{\partial(Lx_j)}{\partial\dot{u}_m}\frac{\partial\dot{u}_m}{\partial x_i} + \frac{\partial(Lx_j)}{\partial\epsilon_{mn}}\frac{\partial\epsilon_{mn}}{\partial x_i} + \left(\frac{\partial(Lx_j)}{\partial x_i}\right)_{\text{expl}}\right]$$

$$= e_{kij}\left[\frac{\partial(Lx_j)}{\partial\dot{u}_m}\frac{\partial\dot{u}_m}{\partial x_i} + \frac{\partial(Lx_j)}{\partial\epsilon^E_{mn}}\frac{\partial\epsilon_{mn}}{\partial x_i} + \left(\frac{\partial(Lx_j)}{\partial x_i}\right)_{\text{expl}}\right]$$

$$= e_{kij}[\rho\dot{u}_m\dot{u}_{m,i}x_j - \sigma_{mn}u_{m,ni}x_j + (L_{,i})_{\text{expl}}x_j] \tag{13}$$

where $(L_{,i})_{\text{expl}}$ is the same as given in equation (8) provided that k is replaced by i. The alternator symbol is denoted by e_{kij}. If the divergence of the stress tensor is introduced, it follows that

$$e_{kij}[(Lx_j)_{,i} - \rho\dot{u}_m\dot{u}_{m,i}x_j + (\sigma_{mn}u_{m,i}x_j)_{,n} - \sigma_{mn,n}u_{m,i}x_j - \sigma_{mn}u_{m,i}x_{j,n}] = e_{kij}(L_{,i})_{\text{expl}}x_j \tag{14}$$

When the equilibrium equations are substituted and the quantity $e_{kij}\sigma_{mi}u_{j,m}$ is added and subtracted to the left side of equation (14), it follows that

$$e_{kij}[(Lx_j)_{,i} + (\sigma_{mn}u_{m,i}x_j)_{,n} - \sigma_{mi}u_{j,m}] = e_{kij}\left[\frac{\partial}{\partial t}(\rho\dot{u}_m u_{m,i}x_j) + (L_{,i})_{\text{expl}}x_j - f_m u_{m,i}x_j - (\sigma_{mi}u_{j,m} - \sigma_{mj}u_{m,i})\right] \tag{15}$$

If the divergence of the stress tensor and the equilibrium equations are then substituted, the desired result is obtained as

$$(e_{kmj}Lx_j + e_{kij}\sigma_{mn}u_{n,i}x_j - e_{kij}\sigma_{mi}u_j)_{,m} = e_{kij}\left[\frac{\partial}{\partial t}(\rho\dot{u}_m u_{m,i}x_j - \rho\dot{u}_i u_j) + (L_{,i})_{\text{expl}}x_j + (f_i u_j - f_m u_{m,i}x_j) - (\sigma_{mi}u_{j,m} - \sigma_{mj}u_{m,i})\right] \tag{16}$$

Equation (16) represents another balance law. The first three quantities in parentheses on the right side account for material inertia, nonhomogeneity and thermal gradients, and body forces, respectively. The last quantity in parentheses vanishes when the material is isotropic, as shown in the Appendix. This equation is not in accord with Fletcher's (1976) equation (3.6) even if nonhomogeneity, thermal graidents, body force, and anisotropy are neglected. Equation (3.6) of Fletcher's paper is flawed, and should have a plus (+) sign preceding the term $e_{imj}x_j u_{l,m}\sigma_{lk}$, rather than a minus (−) sign. Equation (16) appears in integral form as

$$\int_\Gamma e_{kij}(-Lx_j n_i - \sigma_{mn}n_m u_{n,i}x_j + \sigma_{mi}n_m u_j)d\Gamma - \int_\Omega e_{kij}\left[\frac{\partial}{\partial t}(-\rho\dot{u}_m u_{m,i}x_j + \rho\dot{u}_i u_j) - (L_{,i})_{\text{expl}}x_j - (f_i u_j - f_m u_{m,i}x_j) + (\sigma_{mi}u_{j,m} - \sigma_{mj}u_{m,i})\right]d\Omega = 0 \tag{17}$$

Divergence. A third balance law is obtained by considering the divergence of the "Lagrangian moment," i.e.,

$$\nabla\cdot(L\mathbf{x}) = (Lx_k)_{,k}$$

$$= \frac{\partial(Lx_k)}{\partial\dot{u}_i}\frac{\partial\dot{u}_i}{\partial x_k} + \frac{\partial(Lx_k)}{\partial\epsilon_{ij}}\frac{\partial\epsilon_{ij}}{\partial x_k} + \left(\frac{\partial(Lx_k)}{\partial x_k}\right)_{\text{expl}}$$

$$= \frac{\partial(Lx_k)}{\partial\dot{u}_i}\frac{\partial\dot{u}_i}{\partial x_k} + \frac{\partial(Lx_k)}{\partial\epsilon^E_{ij}}\frac{\partial\epsilon_{ij}}{\partial x_k} + \left(\frac{\partial(Lx_k)}{\partial x_k}\right)_{\text{expl}}$$

$$= \rho \dot{u}_i \dot{u}_{i,k} x_k - \sigma_{ij} u_{i,jk} x_k + (L_{,k})_{\mathrm{expl}} x_k + \frac{m}{2}\rho \dot{u}_i \dot{u}_i - \frac{m}{2}\sigma_{ij} d\epsilon^E_{i,j} \quad (18)$$

A parameter m has been introduced which is equal to 3 for three dimensions and 2 for two dimensions. This accounts for a term $x_{k,k}$ which arises from the operations indicated in equation (18). If the divergence of the stress tensor is introduced it follows that

$$(Lx_k)_{,k} + (\sigma_{ij} u_{i,k} x_k)_{,j} - \sigma_{ij,j} u_{i,k} x_k - \sigma_{ij} u_{i,k} x_{k,j} = \rho \dot{u}_i \dot{u}_{i,k} x_k + (L_{,k})_{\mathrm{expl}} x_k + \frac{m}{2}(\rho \dot{u}_i \dot{u}_i - \sigma_{ij}\epsilon^E_{ij}) \quad (19)$$

The equilibrium equations are then substituted resulting in

$$(Lx_j + \sigma_{ij} u_{i,k} x_k)_{,j} = \frac{\partial}{\partial t}(\rho \dot{u}_i u_{i,k} x_k) + \sigma_{ij} u_{i,j} + \frac{m}{2}(\rho \dot{u}_i \dot{u}_i - \sigma_{ij}\epsilon^E_{ij}) + (L_{,k})_{\mathrm{expl}} x_k - f_i u_{i,k} x_k \quad (20)$$

It proves convenient to write equation (20) as

$$(Lx_j + \sigma_{ij} u_{i,k} x_k)_{,j} = \frac{\partial}{\partial t}(\rho \dot{u}_i u_{i,k} x_k) + (-L + \rho \dot{u}_i \dot{u}_i) + (m-1)L + (L_{,k})_{\mathrm{expl}} x_k - f_i u_{i,k} x_k + \sigma_{ij}\alpha_{ij}\Delta\Theta \quad (21)$$

The following two relationships are useful

$$L = \frac{1}{2}\left[\frac{\partial}{\partial t}(\rho \dot{u}_i u_i) - (\sigma_{ij} u_i)_{,j}\right] - \frac{1}{2} f_i u_i + \frac{1}{2}\sigma_{ij}\alpha_{ij}\Delta\Theta \quad (22)$$

$$-L + \rho \dot{u}_i \dot{u}_i = \frac{\partial}{\partial t}[-Lt + \rho t \dot{u}_i \dot{u}_i] - t\left[(\sigma_{ij}\dot{u}_i)_{,j} + f_i \dot{u}_i - \frac{1}{2}\dot{\sigma}_{ij}\alpha_{ij}\Delta\Theta\right] \quad (23)$$

From this follows the desired result

$$\left[Lx_j + \sigma_{ij}\left(u_{i,k}x_k + t\dot{u}_i + \left(\frac{m-1}{2}\right)u_i\right)\right]_{,j} = \frac{\partial}{\partial t}\left[-Lt + \rho \dot{u}_i\left(u_{i,k}x_k + t\dot{u}_i + \left(\frac{m-1}{2}\right)u_i\right)\right] + (L_{,k})_{\mathrm{expl}} x_k - f_i\left(u_{i,k}x_k + t\dot{u}_i + \left(\frac{m-1}{2}\right)u_i\right) + \frac{1}{2}t\dot{\sigma}_{ij}\alpha_{ij}\Delta\Theta + \left(\frac{m+1}{2}\right)\sigma_{ij}\alpha_{ij}\Delta\Theta \quad (24)$$

The term $\dot{\sigma}_{ij}$ may be written alternatively as

$$\dot{\sigma}_{ij} = c_{ijkl}\dot{u}_{k,l} \quad (25)$$

For $m = 3$, equation (24) is in accord with equation (3.5) of the paper by Fletcher (1976) if nonhomogeneity, thermal gradients, and body forces are neglected. Fletcher did not present results for the case $m = 2$. The integral form of equation (24) is

$$\int_\Gamma \left[-Lx_j n_j - \sigma_{ij} n_j\left(u_{i,k}x_k + t\dot{u}_i + \left(\frac{m-1}{2}\right)u_i\right)\right]_{,j} - \int_\Omega \left[\frac{\partial}{\partial t}\left[Lt - \rho \dot{u}_i\left(u_{i,k}x_k + t\dot{u}_i + \left(\frac{m-1}{2}\right)u_i\right)\right] - (L_{,k})_{\mathrm{expl}} x_k + f_i\left(u_{i,k}x_k + t\dot{u}_i + \left(\frac{m-1}{2}\right)u_i\right) - \alpha_{ij}\Delta\Theta\left(\frac{1}{2}tc_{ijkl}\dot{u}_{k,l} + \left(\frac{m+1}{2}\right)\sigma_{ij}\right)\right]d\Omega = 0 \quad (26)$$

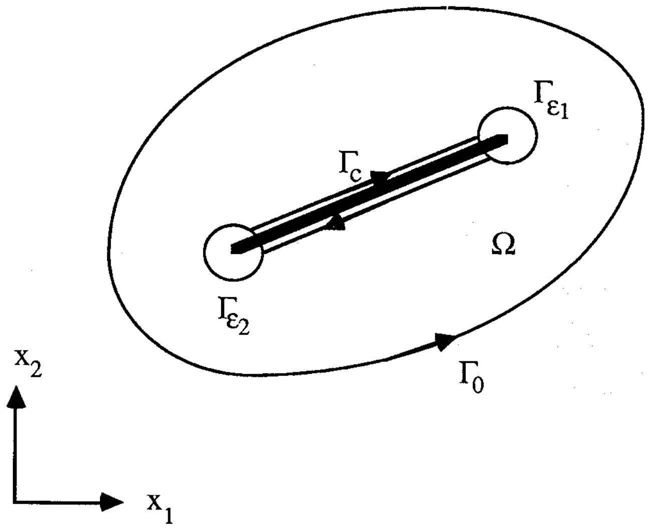

Fig. 1 Schematic of crack and definition of integration paths

3 Energy Release Rates

In this section expressions will be derived for the energy release rates associated with certain crack (defect) motions valid for two-dimensional fracture problems. An extension to three dimensions is straightforward. Figure 1 depicts a crack located in a two-dimensional elastic region. The energy released during three possible crack motions will be studied. These motions are: (*i*) rigid translation such that all points on the crack surfaces move with velocity $\mathbf{v} = v_k\mathbf{e}_k$, where v_k are constants; (*ii*) rigid rotation about the x_3 axis such that points on the crack surface move with velocity $\mathbf{v} = v_k\mathbf{e}_k = -e_{3kl}x_l\omega e_k$, where ω is a positive constant; (*iii*) self-similar expansion along the crack axis such that points on the crack surface move with velocity $\mathbf{v} = v_k\mathbf{e}_k = \alpha x_k\mathbf{e}_k$, where α is a positive constant.

The energy rate balance condition which must hold during any of these crack motions may be stated as follows (see Freund, 1972),

$$P + B = \dot{K} + \dot{U} + F \quad (27)$$

where

$$P = \lim_{\Gamma_\epsilon \to 0} \int_{\Gamma_0} \sigma_{ij} n_j \dot{u}_i d\Gamma \quad (28)$$

$$B = \lim_{\Gamma_\epsilon \to 0} \int_{\Omega(t)} f_i \dot{u}_i d\Omega \quad (29)$$

$$K = \lim_{\Gamma_\epsilon \to 0} \int_{\Omega(t)} \frac{1}{2}\rho \dot{u}_i \dot{u}_i d\Omega \quad (30)$$

$$U = \lim_{\Gamma_\epsilon \to 0} \int_{\Omega(t)} \frac{1}{2}\sigma_{ij}\epsilon^E_{ij} d\Omega \quad (31)$$

The quantities P, B, K, and U are the rate of work of the tractions on Γ_0, the rate of work of the body forces in Ω, the total kinetic energy and elastic strain energies in Ω, respectively. The quantity F is the energy absorption rate on both the crack surfaces and the crack tips measured as a change in energy per unit thickness per unit time. For brevity, the paths Γ_{ϵ_1} and Γ_{ϵ_2} are referred to collectively as Γ_ϵ. The paths Γ_ϵ and Γ_c move rigidly with the crack as it executes the motion described above. Therefore, the position of the paths Γ_ϵ and Γ_c is time-dependent, and the transport theorem must be used to evaluate $\dot{K}$ and $\dot{U}$. Consequently,

$$\dot{K}=\lim_{\Gamma_\epsilon\to 0}\left\{\int_\Omega \rho\ddot{u}_i\dot{u}_i d\Omega+\int_{\Gamma_\epsilon+\Gamma_c}\frac{1}{2}\rho\dot{u}_i\dot{u}_i v_k n_k d\Gamma\right\} \tag{32}$$

$$\dot{U}=\lim_{\Gamma_\epsilon\to 0}\left\{\int_\Omega \sigma_{ij}\dot{u}_{i,j}d\Omega+\int_{\Gamma_\epsilon+\Gamma_c}\frac{1}{2}\sigma_{ij}\epsilon^E_{ij}v_k n_k d\Gamma\right\} \tag{33}$$

After some manipulations it follows that,

$$F=\lim_{\Gamma_\epsilon\to 0}\int_{\Gamma_\epsilon+\Gamma_c}[(T+W)v_k n_k+\sigma_{ij}n_j\dot{u}_i]d\Gamma \tag{34}$$

Recall that T and W are the kinetic and elastic strain energy *densities*, respectively. This expression for the energy release rate will be split into two parts, one having to do with the energy absorbed at the crack tips and another accounting for energy absorbed along the crack surfaces.

$$F=F_{\text{tip}}+F_{\text{sur}} \tag{35}$$

where

$$F_{\text{tip}}=\lim_{\Gamma_\epsilon\to 0}[(T+W)v_k n_k+\sigma_{ij}n_j\dot{u}_i]d\Gamma \tag{36}$$

$$F_{\text{sur}}=\lim_{\Gamma_\epsilon\to 0}\int_{\Gamma_c}[(T+W)v_k n_k]d\Gamma \tag{37}$$

It can be shown that near the tip of an extending crack the field quantities obey the "transport assumption" (Ehrlacher, 1981)

$$\frac{\partial(\ )}{\partial t}=-v_k\frac{\partial(\ )}{\partial x_k} \tag{38}$$

Furthermore, observe that on Γ_c, $n_k^+ = n_k^-$ and $\sigma_{ij}n_j = 0$. After letting $T + W = E$ (kinetic + elastic strain energy density), it follows that

$$F_{\text{tip}}=\lim_{\Gamma_\epsilon\to 0}\int_{\Gamma_\epsilon}[Ev_k n_k-\sigma_{ij}n_j u_{i,k}v_k]d\Gamma \tag{39}$$

$$F_{\text{sur}}=\lim_{\Gamma_\epsilon\to 0}\int_{\Gamma_c}[E^+-E^-]v_k n_k^+ d\Gamma \tag{40}$$

The next step is to evaluate the expression for the crack tip energy release rate given above for the three crack motions discussed earlier.

Case (*i*). v_k = constant

$$\frac{F_{\text{tip}}}{v_k}=G_k^T=\lim_{\Gamma_\epsilon\to 0}\int_{\Gamma_\epsilon}[En_k-\sigma_{ij}n_j u_{i,k}]d\Gamma \tag{41}$$

The energy release rate measured as an energy change per unit translation per unit thickness is denoted G_k^T. This energy release rate can then be written in terms of line integrals around the remote paths Γ_0 and Γ_c, and a domain integral over Ω by applying the divergence theorem to equation (41). Introducing a symbol J_k (k = 1, 2), it follows that

$$J_k=G_k^T=\int_{\Gamma_0}[En_k-\sigma_{ij}n_j u_{i,k}]d\Gamma+\int_{\Gamma_c}[E^+-E^-]n_k^+ d\Gamma-$$

$$\int_\Omega \rho\dot{u}_i\dot{u}_{i,k}-\rho\ddot{u}_i u_{i,k}+f_i u_{i,k}+(E_{,k})_{\text{expl}}]d\Omega \tag{42}$$

This expression may be thought of as an extension to the usual J integral found for crack problems. Equation (42) accounts for inertia, body force, nonhomogeneity, and thermal gradients. For completeness, the expression for $(E_{,k})_{\text{expl}}$ is

$$(E_{,k})_{\text{expl}}=\frac{1}{2}\rho_{,k}\dot{u}_p\dot{u}_p+\frac{1}{2}c_{prst,k}\epsilon^E_{pr}\epsilon^E_{st}$$

$$-\sigma_{pr}\alpha_{pr,k}\Delta\Theta-\sigma_{pr}\alpha_{pr}\Delta\Theta_{,k} \tag{43}$$

Case (*ii*). $v_k = -e_{3kl}x_l\omega$

$$\frac{F_{\text{tip}}}{\omega}=G^R=\lim_{\Gamma_\epsilon\to 0}\int_{\Gamma_\epsilon}e_{3ij}[-Ex_j n_i+\sigma_{mn}n_m u_{n,i}x_j]d\Gamma \tag{44}$$

The energy release rate measured as an energy change per unit angular rotation per unit thickness is denoted G^R. Again, this quantity can be expressed in terms of remote line integrals and a domain integral. Introducing the symbol L_3,

$$L_3=-G^R=\int_{\Gamma_0}e_{3ij}[Ex_j n_i-\sigma_{mn}n_m n_{n,i}x_j+\sigma_{mi}n_m u_j]d\Gamma+$$

$$\int_{\Gamma_c}e_{3ij}[E^+-E^-]x_j n_i^+ d\Gamma-$$

$$\int_\Omega e_{3ij}[\rho\dot{u}_m\dot{u}_{m,i}x_j-\rho\ddot{u}_m u_{m,i}x_j+\rho\ddot{u}_i u_j+(f_m u_{m,i}x_j-f_i u_j)+$$

$$(\sigma_{mi}u_{j,m}-\sigma_{mj}u_{m,i})+(E_{,i})_{\text{expl}}x_j]d\Omega \tag{45}$$

Case (*iii*). $v_k = \alpha x_k$

$$\frac{F_{\text{tip}}}{\alpha}=G^E=\lim_{\Gamma_\epsilon\to 0}\int_{\Gamma_\epsilon}[Ex_k n_k-\sigma_{ij}n_j u_{i,k}x_k]d\Gamma \tag{46}$$

The energy release rate measured as an energy change per unit thickness is denoted G^E. This quantity can be expressed in terms of remote line integrals and a domain integral. Introducing the symbol M,

$$M=G^E=\int_{\Gamma_0}[Ex_k n_k-\sigma_{ij}n_j u_{i,k}x_k]d\Gamma+\int_{\Gamma_c}[E^+-E^-]x_k n_k^+ d\Gamma-$$

$$\int_\Omega[(\rho\dot{u}_i\dot{u}_{i,k}x_k-\rho\ddot{u}_i u_{i,k}x_k+\rho\dot{u}_i\dot{u}_i)+f_i u_{i,k}x_k$$

$$-\sigma_{ij}\alpha_{ij}\Delta\Theta+(E_{,k})_{\text{expl}}x_k]d\Omega \tag{47}$$

A natural question to ask at this point is whether the balance laws derived earlier in Section 2 are connected in some way with the expressions for the energy release rates shown above. After all, Budiansky and Rice (1973) interpreted the integral forms of the conservation laws discovered by Knowles and Sternberg (1972) as energy release rates for the case of elastostatics in the absence of material nonhomogeneity, body forces, anisotropy, and temperature gradients. In the same way, it may be tempting to speculate that the balance laws given by equations (12), (17), and (26) lead to crack tip energy release rate expressions when the path Γ in those expressions is taken as Γ_ϵ and the domain Ω vanishes. If the path Γ is shrunk onto the crack tips, it is clear that the resulting expressions are not directly compatible with the relationships given by equations (41), (44), and (46). For example, equation (12) becomes

$$\int_{\Gamma_\epsilon}[-Ln_k-\sigma_{ij}n_j u_{i,k}]d\Gamma\neq G_k^T \tag{48}$$

One may question whether the domain integral in equation (12) indeed vanishes as $\Gamma\to\Gamma_\epsilon$. This is due to the inverse square root dependence on the distance from the crack tip and the particular angular variations about the crack tip exhibited by the stress and displacement gradient quantities in all fracture problems, independent of inertia, body force, thermal, nonhomogeneity, and anisotropy effects. This argument is made succinctly by Nakamura et al. (1985). The discrepancy between equation (48) and equation (41) is the appearance of the Lagrangian density in the expression derived from the balance law, rather than the quantity E which represents the sum of the kinetic and elastic strain energy densities.

It is instructive to note that if the operations ∇E, $\nabla\times(E\mathbf{x})$, and $\nabla\cdot(E x)$ had been considered, the resulting balance laws would have been in a form readily associated with energy release rates.

$\underline{\nabla E:}$

$$\int_\Gamma [En_k - \sigma_{ij}n_j u_{i,k}]d\Gamma$$

$$- \int_\Omega [\rho\dot{u}_i\dot{u}_{i,k} - \rho\ddot{u}_i u_{i,k} + f_i u_{i,k} + (E_{,k})_{\rm expl}]d\Omega = 0 \qquad (49)$$

$\underline{\nabla \times (E\mathbf{x}):}$

$$\int_\Gamma e_{3ij}[Ex_j n_i - \sigma_{mn}n_m u_{n,i}x_j + \sigma_{mi}n_m u_j]d\Gamma -$$

$$\int_\Omega e_{3ij}[\rho\dot{u}_m\dot{u}_{m,i}x_j - \rho\ddot{u}_m u_{m,i}x_j + \rho\ddot{u}_i u_j + (f_m u_{m,i}x_j - f_i u_j) +$$

$$(\sigma_{mi}u_{j,m} - \sigma_{mj}u_{m,i}) + (E_{,i})_{\rm expl}x_j]d\Omega = 0 \qquad (50)$$

$\underline{\nabla \cdot (E\mathbf{x}):}$

$$\int_\Gamma [Ex_k n_k - \sigma_{ij}n_j u_{i,k}x_k]d\Gamma -$$

$$\int_\Omega [(\rho\dot{u}_i\dot{u}_{i,k}x_k - \rho\ddot{u}_i u_{i,k}x_k + \rho\dot{u}_i\dot{u}_i) + f_i u_{i,k}x_k$$

$$- \sigma_{ij}\alpha_{ij}\Delta\Theta + (E_{,k})_{\rm expl}x_k]d\Omega = 0 \qquad (51)$$

The expressions presented in equations (49)-(51) could have been derived in an alternate manner by eliminating L in favor of E in the balance laws shown earlier.

4 Conclusions

Three of the balance laws presented by Fletcher (1976) have been extended to account for material nonhomogeneity and anisotropy, body force, and thermal gradients. These new balance laws are derived in a systematic manner by subjecting the Lagrangian density to appropriate vector calculus operations. It is pointed out that these balance laws are not directly related to the energy release rates for defect motions. Rather, balance laws expressed in terms of the total (kinetic + elastic strain) energy density are shown to lead to energy release rate expressions.

Acknowledgments

Financial support from the Department of Energy under DOE Grant 12040 to Stanford University is gratefully acknowledged.

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APPENDIX

Assertion: $e_{kij}[\sigma_{mi}u_{j,m} - \sigma_{mj}u_{m,i}] = b_k = 0$ for a linear elastic isotropic material.

Proof: The stress-strain law for a linear elastic isotropic material is

$$\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}) - (3\lambda + 2\mu)\alpha\Delta\Theta\delta_{ij}$$

where δ_{ij} is the Kronecker delta. After substituting this equation in the expression above, the following expression results

$$b_k = \mu e_{kij}(u_{i,m}u_{j,m} - u_{m,j}u_{m,i})$$

The next steps are to multiply both sides of this equation by e_{kst} and apply the identity $e_{kij}e_{kst} = \delta_{is}\delta_{jt} - \delta_{it}\delta_{js}$. It follows that

$$e_{kst}b_k = \mu(\delta_{i3}\delta_{jt} - \delta_{it}\delta_{js})(u_{i,m}u_{j,m} - u_{m,j}u_{m,i})$$

Expanding this expression yields

$$b_k = 0$$

This completes the proof of the assertion above.