References

Kalker, J. J., 1979, "The Computation of Three Dimensional Rolling-Contact with Dry Friction," *Int. J. for Num. Methods in Eng.*, Vol. 14, pp. 1293–1307.

Kishore, N. N., Ghosh, A., Rathore, S. K., and Kishore, P. V., 1994, "Finite Element Analysis of Quasi-Static Contact Problems Using Minimum Dissipation of Energy Principle," ASME JOURNAL OF APPLIED MECHANICS, Vol. 61, pp. 642–648.

Liu, C., and Paul, B., 1989, "Rolling Contact with Friction and Non-Hertzian Pressure Distribution," ASME JOURNAL OF APPLIED MECHANICS, Vol. 56, pp. 814–820.

Moore, D. F., 1975, *The Friction of Pneumatic Tyres*, Elsevier Scientific Pub., Amsterdam.

Padovan, J., Tovichakchaikul, and Zeid, I., 1984, "Finite Element Analysis of Steadily Moving Contact Fields," *Computer and Structures*, Vol. 18, pp. 191–200.

Poritsky, H., 1950, "Stress and deflections of Cylindrical bodies in contact," ASME JOURNAL OF APPLIED MECHANICS, Vol. 18, p. 191.

Zochowski, A., and Myslinski, A., 1991, "Rolling Contact Problems using Quasi-static Variational Formulation," *Computers and Structures*, Vol. 40, pp. 1261–1266.

A Study of Solutions for the Anisotropic Plate Subjected to a Concentrated Force

B. LaMattina,[11,14] **E. C. Klang,**[12,14] **and J. W. Eischen**[13,14]

1 Introduction

The fundamental solution is an essential part of the boundary element method. Bending analysis of plates by the boundary element method requires the use of two fundamental solutions: (1) the displacement field due to a transverse point load, and (2) the displacement field due to a point moment, both for plates of infinite extent. Fundamental solutions for anisotropic plates utilize complex variable theory following groundwork laid by Lekhnitskii (1968). Mossakowski (1955) presented a solution for a point force on an infinite plate using complex parameters of the first kind, and Suchar (1964) presented the solutions for a point force and point moment in terms of complex parameters of the second kind. LaMattina (1997) derived a solution for the point force using the same mapping functions as Mossakowski that is more general than the other two solutions.

One of the objectives of this brief note is to present all three solutions and show how they are applied to the boundary element method formulation. Secondly, differences between these solutions are illustrated and discussed. The implications of these discrepancies is examined for the infinite plate as well as a typical boundary value problem. Another result of this work is related to an arbitrary constant sometimes referred to as the reference radius. This parameter is used to draw a connection between the three solutions. The purpose of the reference radius is discussed and suggestions are made on how to assign its value in numerical implementations of the boundary element method.

[11] Graduate Student, Assoc. Mem. ASME.

[12] Assoc. Professor.

[13] Assoc. Professor, Mem. ASME.

[14] Mechanical and Aerospace Engineering, North Carolina State University, Raleigh, NC 27695.

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, Mar. 20, 1996; final revision, May 23, 1997. Associate Technical Editor: J. N. Reddy.

2 The Fundamental Solution and Boundary Element Method

The basis of the boundary element formulation are two boundary integral equations. The first integral equation is for the point force solution, and the second is for the point moment solution. The direct boundary element method DBEM uses meaningful physical quantities such as deflection, slope, shear forces, and moments in its boundary integral equations. In contrast, the indirect formulation uses functions that do not have a direct physical interpretation. The direct boundary element method integral equations for a generally anisotropic plate are based on the Rayleigh-Green identity. The boundary integral equation for the point force solution is given by Shi and Bezine (1988) as

$$\frac{1}{2} D_{22} w(P) + \int_{\Gamma} \left[V_n^f w - M_n^f \frac{\partial w}{\partial n} + \frac{\partial w^f}{\partial n} M_n - w^f V_n \right] ds + \sum_j^m [|T_n^f w - w^f T_n|]_j = \int_{\Omega} w^f q d\Omega. \quad (1)$$

The fundamental solution for the point load is represented by w^f and the other quantities with the superscript f are computed from the fundamental solution. The boundary element method solution process involves discretizing the boundary and using shape functions similar to finite element analysis. Subsequent to insertion of the fundamental solution and discretization of Eq. (1), the boundary conditions are applied and unknown constants are determined. The deflection at the interior of the plate is determined by another equation similar to Eq. (1).

3 Fundamental Solutions for a Point Load

In this section, the solutions for an infinite anisotropic plate subjected to a concentrated force are presented. The midplane of the plate coincides with the xy-plane and the concentrated force is directed in the positive z direction. The assumptions of classical plate theory are utilized in all the solutions, and the subscript f has been dropped from the fundamental solution ($w = w^f$) for the sake of brevity. The governing differential equation for an anisotropic plate subjected to a concentrated force is given by

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} = P\delta(x - \xi, y - \eta) \quad (2)$$

where the lateral deflection, $w(x, y)$, is the fundamental solution used in Eq. (1). The magnitude of the concentrated force is P, and $\delta(x - \xi, y - \eta)$ is the Dirac delta function. The variables ξ and η give the x and y location of the force, respectively. For simplicity, the point load will be located at the origin of the coordinate system, so that ξ and η are zero.

The characteristic equation is given by

$$D_{22}\mu^4 + 4D_{26}\mu^3 + 2(D_{12} + 2D_{66})\mu^2 + 4D_{16}\mu + D_{11} = 0. \quad (3)$$

The roots μ_j of this equation, also known as complex parameters of the first kind, have been shown by Lekhnitskii to be either complex or purely imaginary.

The solution to Eq. (2) has been presented by three authors for the case when the complex parameters are complex and different. LaMattina and Mossakowski write the solution in terms of the complex parameters of the first kind. Thus, they derive the solution using complex variables defined by

$$z_j = x + \mu_j y \quad \text{for} \quad j = 1, 2. \tag{4}$$

In contrast, Suchar uses complex parameters of the second kind, k_j, which are defined in terms of μ_j as

$$k_j = \frac{1 + i\mu_j}{1 - i\mu_j}. \tag{5}$$

In this case, the xy-plane is mapped into two planes z_{s1} and z_{s2} defined by the following transformation.

$$z_{sj} = (1 + k_j)x + i(1 - k_j)y \quad \text{for} \quad j = 1, 2. \tag{6}$$

The solution presented by LaMattina is given by

$$w = \text{Re} \sum_{j=1}^{2} A_j z_j^2 \ln \left(\frac{z_j}{a_j} \right). \tag{7}$$

The variables a_1, and a_2 are arbitrary constants which are sometimes referred to as reference radii. The constants A_1 and A_2 are given by

$$A_j = \frac{(-1)^j Pi}{2\pi D_{22}(\mu_2 - \mu_1)(\mu_j - \bar{\mu}_1)(\mu_j - \bar{\mu}_2)}. \tag{8}$$

The solution presented by Mossakowski is given by

$$w = \text{Re} \sum_{j=1}^{2} \left[A_j z_j^2 \ln \left(\frac{z_j}{a} \right) - \frac{3}{2} A_j z_j^2 \right] \tag{9}$$

where A_j is the same as Eq. (8).

Suchar writes the fundamental point load solution using the complex variables defined in Eq. (6) as

$$w = \text{Re} \sum_{j=1}^{2} B_j z_{sj}^2 \ln \left(\frac{z_{sj}}{b} \right) \tag{10}$$

where again the constant b is a reference radius.

Here the constants B_1 and B_2 are defined as

$$B_j = \frac{(-1)^j P(1 + k_1)(1 + k_2)(1 + \bar{k}_1)(1 + \bar{k}_2)}{16\pi D_{22}(1 - k_j\bar{k}_1)(1 - k_j\bar{k}_2)(k_2 - k_1)}. \tag{11}$$

4 Comparison of Fundamental Solutions

The three solutions presented above are plotted in Fig. 1 for a $[45/0/90/-45]_s$ carbon/epoxy composite laminate. The lamina properties, corresponding laminate flexural rigidities, and complex parameters are given in Table 1. Figure 1 shows the nondimensionalized deflection, wD_{22}/Ph^2 versus y/h, with the value of x equal to zero. The arbitrary constants a_1, a_2, a, and b are all set equal to one. The three curves have the same general shape but diverge as the distance from the origin increases. It is beneficial to examine the equations more closely in an effort to isolate these differences. Disregarding the reference radii for the moment, a comparison of the author's solution (Eq. (7)) and Mossakowski's solution (Eq. (9)) shows that they differ by the second-order polynomial $(-(3/2)A_j z_j^2)$.

Isolating the difference between the author and Suchar is more complicated since the solutions use different transformation equations. However, manipulation of Eqs. (4)–(6) results in an expression that relates the mapping function z_{sj} to z_j.

$$z_{sj} = (1 + k_j)z_j \tag{12}$$

Another necessary relationship is found by examining the constants A_j and B_j from Eqs. (8) and (11). Consequently,

$$A_j = (1 + k_j)^2 B_j. \tag{13}$$

By substituting Eqs. (12) and (13) into Eq. (10), Suchar's solution becomes

$$w = \text{Re} \sum_{j=1}^{2} \left[A_j z_j^2 \ln \left(\frac{z_j}{b} \right) + A_j z_j^2 \ln (1 + k_j) \right]. \tag{14}$$

Thus, if the reference radii are equal ($a_1 = a_2 = b$), comparing Eqs. (7) and (14) reveals that the solution derived by Suchar differs from the author's solution by the polynomial $[A_j z_j^2 \ln (1 + k_j)]$.

In order to explain the discrepancy in results, the solution process for boundary value problems must be examined from the beginning. A well-posed boundary value problem consists of the governing equation as well as the boundary conditions. To obtain a unique solution one must satisfy both the governing equation and the boundary conditions. The general solution to the governing differential equation consists of two parts. The first part satisfies the homogeneous differential equation and is often referred to as the complimentary solution. The complimentary solution usually contains unknown constants. The second part of the general solution is usually called a particular solution and is any solution that satisfies the nonhomogeneous governing equation. The particular solution does not contain unknown constants and is not unique. Particular solutions may differ from one another by a multiple of any of the terms of the complimentary solution. The general solution is a linear combination of the complimentary and particular solutions. Once the general solution is formed, the boundary conditions are applied, the unknown constants are determined, and the solution becomes unique.

The fundamental solutions presented in this note are particular solutions for a plate subjected to a concentrated force. They satisfy the nonhomogeneous Eq. (2) and were not subject to any boundary conditions. Therefore, if all of these solutions are valid they should only differ by some multiple of a term of the complimentary solution. It has been shown that both Mossakowski's and Suchar's solutions differ from

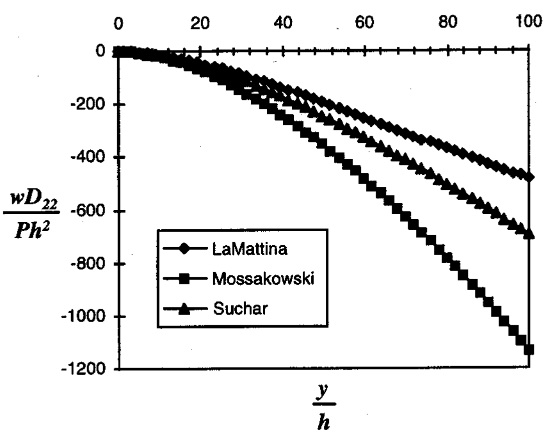

Fig. 1 Nondimensionalized deflection for the three solutions

Table 1 Composite properties and complex parameters

Lamina Properties, Composite Flexural Rigidities, and Complex Parameters
E_1=159.0 GPa E_2=10.0 GPa G_{12}=5.9 GPa v_{12}=0.31
D_{11}=11.3 Nm D_{12}=3.4 Nm D_{16}=3.0 Nm D_{22}=7.2 Nm D_{26}=3.0 Nm D_{66}=3.8 Nm
μ_1=-0.134 + 0.903i μ_2=-0.706 + 1.171i k_1=0.046 - 0.074i k_2=-0.167 - 0.271i
Lamina Thickness=0.15mm

the author's by a multiple of the polynomial z_j^2. Thus, consider the function

$$w_{c1} = C_j z_j^2 \tag{15}$$

where C_j is any constant. Substituting Eq. (4) into (15) gives

$$w_{c1} = C_j(x + \mu_j y)^2. \tag{16}$$

The homogeneous version of the governing Eq. (2) is clearly satisfied by Eq. (16). Hence, this term is also a term of the complimentary solution and will therefore be inconsequential when considering a boundary value problem.

5 The Reference Radius

In this section, the consequences and purpose of the reference radius is examined more closely. Almost all the plate bending solutions that contain the natural log term include the reference radius (Lekhnitskii, 1968; Mossakowski, 1955; Suchar, 1964; Timoshenko et al., 1959). However, none of these publications state anything about this constant except that it is arbitrary.

Examination of any of the fundamental solutions shows that they will differ for different values of the reference radius. Thus, the obvious question one may ask is "How can this term be considered arbitrary if it changes the solution"? This is a valid question which can be answered by rewriting the solutions in a slightly different form. For example, Eq. (7) can be rewritten as

$$w = \mathrm{Re} \sum_{j=1}^{2} [A_j z_j^2 \ln z_j - A_j z_j^2 \ln a_j]. \tag{17}$$

Scrutiny of Eq. (17) reveals that the inclusion of the reference radius is equivalent to including the polynomial term $[-A_j z_j^2 \ln a_j]$. However, as is pointed out in the previous section, any second-order polynomial of the form $[C_j z_j^2]$ is a complimentary solution. Thus, its incorporation in the particular solution will have no effect on the uniqueness of a boundary value problem.

In fact, by a prudent choice of the reference radii the solution presented by the author can be made to match the solutions by Mossakowski or Suchar. By setting Eq. (7) equal to Eq. (9) and solving for a_j gives a relation for the reference radii that makes the solutions by the author and Mossakowski identical. The author's solution can be made to match Suchar's by equating Eq. (7) and (10). The relationships between the reference radii are summarized in Table 2. For the numerical example presented in Section 4 the values of the reference radii are given in Table 3. Tables 2 and 3 show that the solution presented by the author can be made to match either of the other solutions by choosing the proper reference radius. However, it is not possible to make the solutions by Mossakowski and Suchar match each other's by choosing a reference radius. This is because these solutions only contain one reference radius, whereas the author's solution has two.

6 Discussion

Three solutions for the generally anisotropic plate subjected to a concentrated force are presented. This paper illustrates that each of these solutions are different from one another. It also points out that they are particular solutions and that the lack of uniqueness among them is attributed to the addition of a term from the complementary solution. Hence, any of these three solutions given by Eqs. (7), (9), or (10), respectively, can be used as the fundamental solution for the boundary element method.

Perhaps the most obvious reason for including the reference radius is for dimensional consistency of the equations. By including a reference radius with units of length, the log term becomes dimensionless, and the units on the left and right sides of the equations will match. Another positive side effect of the reference radius comes into play for numerical solution methods, such as the boundary element method. The reference radius provides a variable that can be assigned a value that will aid in avoiding ill-conditioned equations. In the indirect boundary element method formulation presented by Wu and Alterio (1981), a range of values for the reference radius is given for which their results are accurate. The author believes that avoiding ill-conditioned equations is the reason for choosing these values. In the boundary element method it is sufficient to choose the reference radius to have the same order of magnitude as the largest plate dimension.

As illustrated in Eq. (7), it is not necessary to use the same value for the reference radius in both terms of the fundamental solution as indicated in the papers by Mossakowski and Suchar. Choosing different values will not effect the end result for a boundary value problem. Secondly, it is not necessary to restrict the reference radii to only real numbers, they can be complex. In fact, this is the case when the reference radii are defined according to Table 3.

Table 2 Connection between the reference radii

Mossakowski ⇒ LaMattina :	$a_j = ae^{1/2}$,	a = arbitrary
Suchar ⇒ LaMattina :	$a_j = \dfrac{b}{1+k_j}$,	b = arbitrary

Table 3 Values of the reference radii to match solutions for the [45/0/90/-45]$_s$ composite

Mossakowski ⇒ LaMattina
$a_1 = a_2 = e^{1/2}$ m, $a = 1$ m
Suchar ⇒ LaMattina
$a_1 = 0.9517 + 0.0671i$ m, $a_2 = 1.0855 + 0.3531i$ m, $b = 1$ m

Acknowledgment

This work was funded in part by NASA grant NAGW-1331.

References

LaMattina, B., 1997, "Anisotropic Plate Bending Analysis Using Complex Variables Methods," Ph.D. dissertation, North Carolina State University, Raleigh, NC.

Lekhnitskii, S. G., 1968, *Anisotropic Plates,* Gordon and Breach, New York.

Mossakowski, J., 1955, "Singular Solutions of Anisotropic Plates," *Arch. Mech. Stos.,* Vol. 7, No. 1, pp. 97–110 (in Polish).

Shi, G., and Bézine, G., 1988, "A General Boundary Integral Formulation for the Anisotropic Plate Bending Problems," *Journal of Composite Materials,* Vol. 22, pp. 694–716.

Suchar, M., 1964, "On Singular Solutions in the Theory of Anisotropic Plates," *Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Techniques,* Vol. 12, pp. 29–38.

Timoshenko, S., and Woinowsky-Krieger, S., 1959, *Theory of Plates and Shells,* McGraw-Hill, New York, pp. 325–328.

Wu, B. C., and Altiero, N. J., 1981, "A New Numerical Method for the Analysis of Anisotropic Thin-Plate Bending Problems," *Comp. Meth. Appl. Mech. and Eng.,* Vol. 25, pp. 343–353.