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Harmonic Wave Propagation in Nonhomogeneous Layered Composites

A new method for analyzing plane wave propagation in a periodically layered, elastic, nonhomogeneous composite body is proposed. The nonhomogeneity considered is a variation of the material properties within each composite layer. Results from probability theory are used to arrive at the two fundamental solutions of the governing second order ordinary differential equations. Floquet's wave theory is combined with a Wronskian formula to yield the dispersion relationship for this nonhomogeneous composite. Numerical results show that the presence of material nonhomogeneity affects the range of frequencies which can pass through the composite unattenuated.

1 Introduction

Problems of wave propagation in layered elastic composites have attracted a great deal of attention from researchers during recent years [1–6]. Several studies [7–9] have used Floquet's theory for one-dimensional wave propagation or Bloch's theory for three-dimensional wave propagation. These investigations have shown that when the wavelength of a harmonic wave is comparable to the characteristic length of the composite layers, successive reflection and refraction of the waves from the interfaces between layers leads to a significant dispersive effect. Such phenomena cannot be predicted by so-called "effective modulus" theories. For anti-plane or plane strain waves, the dispersion relationship can be interpreted geometrically as a surface in the wave number-frequency space. The important feature that was discovered is the presence of pass bands and stop bands, i.e., regions in the frequency spectrum where harmonic waves are either propagated freely or attenuated, respectively. The curves on the surface which define the boundary between the pass bands and stop bands divide the surface into so-called Brillouin zones.

The analyses made by Delph, Herrmann, and Kau [7–9] and by other researchers were performed with the assumption that the material properties within each layer of the composite were homogeneous. However, considering realistic manufacturing processes and/or naturally occurring variations it may not be reasonable to expect a uniform distribution of the elastic constants and mass density throughout each composite layer. It is the purpose of this paper to present a general method to analyze the situtíon in which the cells in the periodically layered composite structure are nonhomogeneous, i.e., the elastic constants and mass density depend on the spatial coordinates within each layer.

This method is based on a procedure of representing the solution of the governing second order ordinary differential equations by means of a technique taken from probability theory. Combining Floquet's wave theory with properties of a special Wronskian formula, the dispersion relationship for wave propagation in certain nonhomogeneous composites is derived. Numerical calculations pertaining to the dispersion relationship for nonhomogeneous composites have shown that the presence of a material nonhomogeneity within each layer of the composite alters the width of the stop band and affects the dissipative characteristics of the medium.

2 Derivation of the Dispersion Relationship

The system under consideration consists of an infinite sequence of two alternating layers, each of which are taken to be nonhomogeneous and elastic. Perfect bonding is assumed between the adjoining layers. A unit cell is defined as the union of any two adjacent layers. As shown in Fig. 1, the two lamellae of the *N-th* unit cell have variable Lamé moduli $\{\lambda_m(x),\ \mu_m(x)\}$, $\{\lambda_f(x),\ \mu_f(x)\}$, variable mass densities $\{\rho_m(x),\ \rho_f(x)\}$, and thicknesses $\{2h_m,\ 2h_f\}$, where the subscripts m and f refer to "matrix" and "fiber" layers, respectively.

Let u, v, and w be the three Cartesian components of the displacement vector in the x, y and z directions, respectively. The layers lie in the y-z plane. Consideration will be given only to waves propagating in a direction normal to the layers. For a one-dimensional longitudinal strain wave propagtaing in the x-direction, only the u component of displacement is nonzero. Therefore, we take

$$u = u(x,t) \quad v = w = 0 \tag{2.1}$$

where the function $u(x,t)$ satisfies the equation of motion

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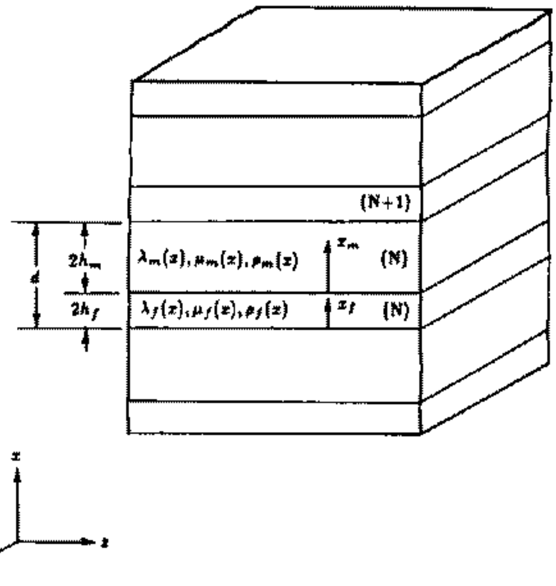

Fig. 1 Geometry of nonhomogeneous layered composite

$$\rho(x)\,\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left(D(x)\,\frac{\partial u}{\partial x}\right) \tag{2.2}$$

where $D(x) = \lambda(x) + 2\mu(x)$. The material properties $D(x)$ and $\rho(x)$ satisfy periodicity conditions

$$D(x+d)=D(x) \quad \rho(x+d)=\rho(x) \tag{2.3$_{1\text{-}2}$}$$

where $d = 2(h_m + h_f)$ represents the thickness of one unit cell, while

$$D(x)=\begin{cases} D_m(x), & \text{for } matrix \\ D_f(x), & \text{for } fiber \end{cases} \tag{2.4}$$

and

$$\rho(x)=\begin{cases} \rho_m(x), & \text{for } matrix \\ \rho_f(x), & \text{for } fiber \end{cases} \tag{2.5}$$

The global coordinate x is replaced with two local coordinates, x_m and x_f, in the N-th unit cell, as shown in Fig. 1. Thus, (2.2) holds in each layer

$$\rho_m(x_m)\,\frac{\partial^2 u_m}{\partial t^2}=\frac{\partial}{\partial x_m}\left(D_m(x_m)\,\frac{\partial u_m}{\partial x_m}\right) \quad (0\le x_m \le 2h_m) \tag{2.6$_1$}$$

$$\rho_f(x_f)\,\frac{\partial^2 u_f}{\partial t^2}=\frac{\partial}{\partial x_f}\left(D_f(x_f)\,\frac{\partial u_f}{\partial x_f}\right) \quad (0\le x_f \le 2h_f) \tag{2.6$_2$}$$

The solutions of $(2.6)_{1\text{-}2}$ are taken in the form of harmonic plane waves as

$$u_m=\left[\frac{D_m(0)}{D_m(x_m)}\right]^{1/2} U_m(x_m)e^{-i\omega t} \tag{2.7$_1$}$$

$$u_f=\left[\frac{D_f(0)}{D_f(x_f)}\right]^{1/2} U_f(x_f)e^{-i\omega t} \tag{2.7$_2$}$$

where ω is the circular frequency measured in radians per second, $U_m(x_m)$ and $U_f(x_f)$ are two unknown functions representing the amplitude of vibration. The longitudinal wave speeds will be denoted by $c_m = \sqrt{D_m/\rho_m}$ and $c_f = \sqrt{D_f/\rho_f}$.

We next introduce nondimensional parameters and nondimensional dependent and independent variables according to

$$\bar{x}_f=\frac{x_f}{2h_f} \quad \bar{x}_m=\frac{x_m}{2h_m} \quad \bar{U}_f=\frac{U_f}{2h_f} \quad \bar{U}_m=\frac{U_m}{2h_m}$$

$$\Omega=\frac{2\omega h_f}{\pi c_{fo}} \quad \tau=\frac{\pi c_{fo}}{2h_f}t \quad \bar{\sigma}_f=\frac{\sigma_f}{D_f(0)} \quad \bar{\sigma}_m=\frac{\sigma_m}{D_m(0)} \tag{2.8}$$

$$\bar{c}_f=\frac{c_f}{c_{fo}} \quad \bar{c}_m=\frac{c_m}{c_{mo}} \quad \bar{D}_f(\bar{x}_f)=\frac{D_f(x_f)}{D_f(0)}$$

$$\bar{D}_m(\bar{x}_m)=\frac{D_m(x_m)}{D_m(0)}$$

where $c_{mo} = \sqrt{D_m(0)/\rho_m(0)}$ and $c_{fo} = \sqrt{D_f(0)/\rho_f(0)}$ represent the longitudinal wave speeds at the interface between the matrix and fiber, respectively. The stresses in the matrix and fiber layers are indicated by σ_m and σ_f, respectively. Using these nondimensional quantities and substituting $(2.7)_{1\text{-}2}$ into $(2.6)_{1\text{-}2}$, the equations of motion are reduced to a system of second order Sturm-Liouville ordinary differential equations

$$\frac{d^2\bar{U}_m}{d\bar{x}_m^2}+V_m(\bar{x}_m,\Omega)\bar{U}_m=0 \tag{2.9$_1$}$$

$$\frac{d^2\bar{U}_f}{d\bar{x}_f^2}+V_f(\bar{x}_f,\Omega)\bar{U}_f=0 \tag{2.9$_2$}$$

with variable coefficients given by

$$V_m(\bar{x}_m,\Omega)=\frac{\Omega^2\pi^{*2}}{\bar{c}_m^2(\bar{x}_m)}-\frac{1}{2}\frac{d}{d\bar{x}_m}\left(\frac{1}{\bar{D}_m}\frac{d\bar{D}_m}{d\bar{x}_m}\right)-\frac{1}{4}\left(\frac{1}{\bar{D}_m}\frac{d\bar{D}_m}{d\bar{x}_m}\right)^2 \tag{2.10$_1$}$$

$$V_f(\bar{x}_f,\Omega)=\frac{\Omega^2\pi^2}{\bar{c}_f^2(\bar{x}_f)}-\frac{1}{2}\frac{d}{d\bar{x}_f}\left(\frac{1}{\bar{D}_f}\frac{d\bar{D}_f}{d\bar{x}_f}\right)-\frac{1}{4}\left(\frac{1}{\bar{D}_f}\frac{d\bar{D}_f}{d\bar{x}_f}\right)^2 \tag{2.10$_2$}$$

where $\pi^* = \pi\dfrac{h_m c_{fo}}{h_f c_{mo}}$.

For convenience, we now revert to mathematical notations introduced initially to indicate the corresponding nondimensional quantities, thereby dropping the barred notation. It is hoped that this will not confuse the readers. Furthermore, since both independent variables $\bar{x}_m$ and $\bar{x}_f$ vary between (0, 1), there is no need to distinguish between them. In the following derivation, we will let x stand for both $\bar{x}_m$ and $\bar{x}_f$.

Let U_{mi} (x, Ω) and $U_{fi}(x, \Omega)$ $(i = 1, 2)$ be the two fundamental solutions of $(2.9)_{1\text{-}2}$, respectively, satisfying the boundary conditions

$$U_{m1}(0,\Omega)=U_{m2}(1,\Omega)=U_{f1}(0,\Omega)=U_{f2}(1,\Omega)=1 \tag{2.11$_1$}$$

$$U_{m1}(1,\Omega)=U_{m2}(0,\Omega)=U_{f1}(1,\Omega)=U_{f2}(0,\Omega)=0 \tag{2.11$_2$}$$

Since the coefficients V_m (x, Ω) and V_f (x, Ω) vary with x, determination of analytical expressions for the fundamental solutions is not a simple task. However, when Ω is not an eigenvalue of $(2.9)_{1\text{-}2}$, a method recently developed by Chung [10] which uses probability theory allows the solutions U_{mi} (x, Ω) and U_{fi} (x, Ω) to be expressed in closed-form as

$$U_{m1}(x,\Omega)=\int_{[B_\tau=0]}\exp\left(\int_0^\tau V_m(B_t,\Omega)dt\right)dp^x$$

$$U_{m2}(x,\Omega)=\int_{[B_\tau=1]}\exp\left(\int_0^\tau V_m(B_t,\Omega)dt\right)dp^x \tag{2.12$_1$}$$

and

$$U_{f1}(x,\Omega)=\int_{[B_\tau=0]}\exp\left(\int_0^\tau V_f(B_t,\Omega)dt\right)dp^x$$

$$U_{f2}(x,\Omega)=\int_{[B_\tau=1]}\exp\left(\int_0^\tau V_f(B_t,\Omega)dt\right)dp^x \tag{2.12$_2$}$$

where $[B_\tau;\ \tau>0]$ stands for a Brownian motion process, τ means the first exit time from the interval (0, 1), and p^x is the probability distribution of the process $[B_t;\ t>0]$ starting at x. In fact, using a random walk instead of Brownian motion, a second order finite difference scheme can be formulated to calculate the fundamental solutions $(2.12)_{1-2}$ very accurately. This procedure will be illustrated in a separate paper.

Complex forms of the fundamental solutions are given in terms of $U_{mi}\ (x, \Omega)$ and $U_{fi}\ (x, \Omega)$ by

$$\mathcal{U}_{m1}(x,\Omega)=U_{m1}(x,\Omega)+iU_{m2}(x,\Omega) \qquad (2.13)_1$$

$$\mathcal{U}_{m2}(x,\Omega)=U_{m1}(x,\Omega)-iU_{m2}(x,\Omega) \qquad (2.13)_2$$

$$\mathcal{U}_{f1}(x,\Omega)=U_{f1}(x,\Omega)+iU_{f2}(x,\Omega) \qquad (2.14)_1$$

$$\mathcal{U}_{f2}(x,\Omega)=U_{f1}(x,\Omega)-iU_{f2}(x,\Omega) \qquad (2.14)_2$$

They must satisfy the boundary conditions

$$\mathcal{U}_{m1}(0,\Omega)=\mathcal{U}_{m2}(0,\Omega)=\mathcal{U}_{f1}(0,\Omega)=\mathcal{U}_{f2}(0,\Omega)=1 \qquad (2.15)$$

and

$$\mathcal{U}_{m1}(1,\Omega)=\mathcal{U}_{f1}(1,\Omega)=i, \qquad \mathcal{U}_{m2}(1,\Omega)=\mathcal{U}_{f2}(1,\Omega)=-i \qquad (2.16)_{1-2}$$

The general solution for the displacement in the matrix layer of the N-th unit cell can be written in the form

$$u_m(x,\Omega)=\frac{1}{\sqrt{D_m(x)}}\,[A_m\mathcal{U}_{m1}(x,\Omega)+B_m\mathcal{U}_{m2}(x,\Omega)]e^{-i\Omega\tau} \qquad (2.17)$$

where A_m and B_m are nontrivial complex constants yet to be determined. Using the stress-strain relations, the stress in the matrix layer of the N-th unit cell is given by

$$\sigma_m=D_m\frac{\partial u_m}{\partial x}=\frac{1}{\sqrt{D_m(x)}}\,[A_m\sigma_{m1}+B_m\sigma_{m2}]e^{-i\Omega\tau} \qquad (2.18)$$

where

$$\sigma_{m1}=D_m\mathcal{U}'_{m1}(x,\Omega)-\frac{1}{2}\,D'_m\mathcal{U}_{m1}(x,\Omega) \qquad (2.19)_1$$

$$\sigma_{m2}=D_m\mathcal{U}'_{m2}(x,\Omega)-\frac{1}{2}\,D'_m\mathcal{U}_{m2}(x,\Omega) \qquad (2.19)_2$$

The prime represents a derivative of the associated quantity with respect to x. Similarly, in the fiber layer of the N-th unit cell, the displacement and stress take the forms

$$u_f(x,\Omega)=\frac{1}{\sqrt{D_f(x)}}\,[A_f\mathcal{U}_{f1}(x,\Omega)+B_f\mathcal{U}_{f2}(x,\Omega)]e^{-i\Omega\tau} \qquad (2.20)$$

$$\sigma_f=D_f\frac{\partial u_f}{\partial x}=\frac{1}{\sqrt{D_f(x)}}\,(A_f\sigma_{f1}+B_f\sigma_{f2})e^{-i\Omega\tau} \qquad (2.21)$$

where

$$\sigma_{f1}=D_f\mathcal{U}'_{f1}(x,\Omega)-\frac{1}{2}\,D'_f\mathcal{U}_{f1}(x,\Omega) \qquad (2.22)_1$$

$$\sigma_{f2}=D_f\mathcal{U}'_{f2}(x,\Omega)-\frac{1}{2}\,D'_f\mathcal{U}_{f2}(x,\Omega) \qquad (2.22)_2$$

To complete specification of the problem, continuity of displacement and traction must be enforced at the interface between matrix and fiber layers, which leads to

$$u_f(1,\Omega)=\xi u_m(0,\Omega) \quad \sigma_f(1,\Omega)=\eta\sigma_m(0,\Omega) \qquad (2.23)_{1-2}$$

and

$$u_f^*(0,\Omega)=\xi u_m(1,\Omega) \quad \sigma_f^*(0,\Omega)=\eta\sigma_m(1,\Omega) \qquad (2.24)_{1-2}$$

where $\xi=h_m/h_f$ and $\eta=\rho_{mo}c_{mo}^2/\rho_{fo}c_{fo}^2$. Moreover, u_f^* and σ_f^* represent the displacement and stress in the fiber layer of the $(N+1)$-th unit cell, respectively.

According to Floquet's one-dimensional wave theory, $(2.7)_2$ with its periodic variable coefficients admits quasi-periodic recurrence relations for the displacement and stress between two adjacent cell units as follows

$$u_f^*(0,\Omega)=u_f(0,\Omega)e^{ikd} \quad \sigma_f^*(0,\Omega)=\sigma_f(0,\Omega)e^{ikd} \qquad (2.25)_{1-2}$$

where $k=k_1+ik_2$ is the complex Floquet wave number. Combining continuity conditions $(2.24)_{1-2}$ with $(2.25)_{1-2}$, we obtain

$$u_f(0,\Omega)e^{ikd}=\xi u_m(1,\Omega) \quad \sigma_f(0,\Omega)e^{ikd}=\eta\sigma_m(1,\Omega) \qquad (2.26)_{1-2}$$

Substitution of expressions (2.18) and (2.21) for the stresses σ_m and σ_f into $(2.23)_2$ and $(2.26)_2$, respectively, yields a set of four homogeneous algebraic equations from which the unknown constants A_m, B_m, A_f, and B_f can be determined. A nontrivial solution for these constants exists if the corresponding determinant of the matrix of coefficients vanishes. Setting the determinant equal to zero leads to the following dispersion equation

$$\begin{vmatrix} i\epsilon & -i\epsilon & -1 & -1 \\ \epsilon e^{ikd} & \epsilon e^{ikd} & -i & i \\ \mathcal{U}'_{m1}(1,\Omega)+i\beta_m & \mathcal{U}'_{m2}(1,\Omega)-i\beta_m & -\mathcal{U}'_{f1}(0,\Omega)+\beta_f & -\mathcal{U}'_{f2}(0,\Omega)+\beta_f \\ (\mathcal{U}'_{m1}(0,\Omega)-\beta_m)e^{ikd} & (\mathcal{U}'_{m2}(0,\Omega)-\beta_m)e^{ikd} & -\mathcal{U}'_{f1}(1,\Omega)-i\beta_f & -\mathcal{U}'_{f2}(1,\Omega)+i\beta_f \end{vmatrix}=0 \qquad (2.27)$$

where three new parameters have been introduced as

$$\epsilon=\frac{\eta}{\xi} \quad \beta_f=\frac{1}{2}\,D'_f(0) \quad \beta_m=\frac{1}{2}\,D'_m(0) \qquad (2.28)_{1-3}$$

We see from (2.27) that the dispersion relationship depends on derivatives of the complex fundamental solutions $\mathcal{U}_{mi}\,(x, \Omega)$ and $\mathcal{U}_{fi}\ (x, \Omega)$ evaluated at the end points $x=0$ and $x=1$. These derivatives are related to the corresponding derivatives of the real valued fundamental solutions as follows

$$\begin{aligned}
\mathcal{U}'_{m1}(0,\Omega)&=U'_{m1}(0,\Omega)+iU'_{m2}(0,\Omega) & \mathcal{U}'_{m2}(0,\Omega)&=U'_{m1}(0,\Omega)-iU'_{m2}(0,\Omega)\\
\mathcal{U}'_{m1}(1,\Omega)&=U'_{m1}(1,\Omega)+iU'_{m2}(1,\Omega) & \mathcal{U}'_{m2}(1,\Omega)&=U'_{m1}(1,\Omega)-iU'_{m2}(1,\Omega)\\
\mathcal{U}'_{f1}(0,\Omega)&=U'_{f1}(0,\Omega)+iU'_{f2}(0,\Omega) & \mathcal{U}'_{f2}(0,\Omega)&=U'_{f1}(0,\Omega)-iU'_{f2}(0,\Omega)\\
\mathcal{U}'_{f1}(1,\Omega)&=U'_{f1}(1,\Omega)+iU'_{f2}(1,\Omega) & \mathcal{U}'_{f2}(1,\Omega)&=U'_{f1}(1,\Omega)-iU'_{f2}(1,\Omega)
\end{aligned} \qquad (2.29)$$

Before expanding the determinant (2.27), we will examine some important features of the fundamental solutions $U_{mi}\,(x, \Omega)$ and $U_{fi}\,(x, \Omega)$. Henceforth, we assume that the material properties in each layer of any unit cell are symmetric with respect to the midplane of that layer.

Proposition 1. If the coefficient $V_f(x, \Omega)$ in $(2.9)_2$ is symmetric with respect to the midplane $x = 0.5$ in the interval (0, 1), such that

$$V_f(x,\Omega) = V_f(1-x,\Omega) \qquad (2.30)$$

then the derivatives of the fundamental solutions $U'_{f1}(x, \Omega)$ and $U'_{f2}(x, \Omega)$ at both ends $x = 0$ and $x = 1$ satisfy

$$U'_{f1}(1,\Omega) + U'_{f2}(0,\Omega) = 0 \quad U'_{f1}(0,\Omega) + U'_{f2}(1,\Omega) = 0 \qquad (2.31)_{1\text{-}2}$$

Proof: The Wronskian $W_f[U_{f1}, U_{f2}]$ of the fundamental solutions $U_{f1}(x, \Omega)$ and $U_{f2}(x, \Omega)$ is defined as

$$W_f[U_{f1}, U_{f2}] \equiv \begin{vmatrix} U_{f1}(x,\Omega) & U_{f2}(x,\Omega) \\ U'_{f1}(x,\Omega) & U'_{f2}(x,\Omega) \end{vmatrix} \qquad (2.32)$$

Furthermore, due to the symmetry of $V_f(x, \Omega)$ and based on the basic behavior of the Wronskian, we have

$$\frac{dW_f}{dx} = 0 \qquad (0 \le x \le 1) \qquad (2.33)$$

Thus

$$W_f(0,\Omega) = W_f(1,\Omega) \qquad (2.34)$$

or written in expanded form

$$\begin{vmatrix} U_{f1}(0,\Omega) & U_{f2}(0,\Omega) \\ U'_{f1}(0,\Omega) & U'_{f2}(0,\Omega) \end{vmatrix} = \begin{vmatrix} U_{f1}(1,\Omega) & U_{f2}(1,\Omega) \\ U'_{f1}(1,\Omega) & U'_{f2}(1,\Omega) \end{vmatrix} \qquad (2.35)$$

Therefore, according to the boundary conditions $(2.11)_{1\text{-}2}$, (2.35) is equivalent to $(2.31)_1$. Using the symmetry condition (2.30), $(2.31)_2$ is easily confirmed. The same argument follows for the behavior in the matrix layer.

Making use of (2.29) and (2.31), the dispersion relationship (2.27) reduces to the simple form[1-2]

$$e^{2ikd} + F(\Omega)e^{ikd} + 1 = 0 \qquad (2.36)$$

where $F(\Omega)$ is called the "spectrum function" and is given by

$$F(\Omega) = \frac{\left(U'_{f1}(1,\Omega)\right)^2 + \epsilon^2\left(U'_{m1}(1,\Omega)\right)^2 - \left(U'_{f1}(0,\Omega) + \epsilon U'_{m1}(0,\Omega) - \beta_f - \epsilon\beta_m\right)^2}{\epsilon U'_{f1}(1,\Omega)U'_{m1}(1,\Omega)} \qquad (2.37)$$

The derivatives of the first fundamental solutions $U_{m1}(x, \Omega)$ and $U_{f1}(x, \Omega)$ must be known at the ends $x = 0$ and $x = 1$ in order to determine the function $F(\Omega)$. For this reason, we present two basic properties of these derivatives.

Proposition 2. If Ω is not an eigenvalue of $(2.9)_{1\text{-}2}$, then the derivatives $U'_{m1}(1, \Omega)$ and $U'_{f1}(1, \Omega)$ must not vanish. This will insure that $F(\Omega)$ remains bounded.

Proof: If the independent and dependent variables are transformed according to

$$x^* = 1 - x \quad U^*_{f1}(x^*,\Omega) = U_{f1}(1-x,\Omega) \qquad (2.38)$$

and the symmetry condition (2.30) is used, $(2.9)_2$ takes the form

$$\frac{d^2U^*_f}{dx^{*2}_f} + V_f(x^*,\Omega)U^*_f = 0 \qquad (2.39)$$

On the other hand, according to $(2.11)_{1\text{-}2}$ and (2.38), we find that

$$U_{f1}(1,\Omega) = 0 \Rightarrow U^*_{f1}(0,\Omega) = 0 \qquad (2.40)$$

Therefore, if $U'_{f1}(1,\Omega)$ is to vanish, according to (2.38), $U'^*_{f1}(0,\Omega)$ must vanish also, i.e.,

$$U'^*_{f1}(0,\Omega) = 0 \qquad (2.41)$$

The existence and uniqueness theorem states that if Ω is not an eigenvalue of the ordinary differential equation (2.39), and homogeneous boundary conditions (2.40) and (2.41) are posed, then (2.39) has only a trivial solution

$$U^*_{f1}(x^*,\Omega) = 0 \qquad (2.42)$$

Obviously

$$U^*_{f1}(1,\Omega) = 0 \Rightarrow U_{f1}(0,\Omega) = 0 \qquad (2.43)$$

However, this conclusion is in contradiction to the original assumption $(2.11)_1$, and therefore $U'_{f1}(1, \Omega)$ must not vanish. The same argument can be made concerning $U'_{m1}(1, \Omega)$.

Proposition 3. If Ω is not an eigenvalue of $(2.9)_1$ the derivatives $U'_{f1}(0, \Omega)$ and $U'_{f1}(1, \Omega)$ can be expressed in the form

$$U'_{f1}(0,\Omega) = \frac{b_2u_1(1,\Omega) - b_1u_2(1,\Omega)}{a_2u_1(1,\Omega) - a_1u_2(1,\Omega)} \qquad (2.44)_1$$

$$U'_{f1}(1,\Omega) = \frac{a_2b_1 - a_1b_2}{a_2u_1(1,\Omega) - a_1u_2(1,\Omega)} \qquad (2.44)_2$$

where the functions $u_i(x, \Omega)$ $(i = 1, 2)$ are solutions of the following initial value problems

$$\frac{d^2u_i}{dx^2} + V_f(x,\Omega)u_i = 0 \quad (i = 1,2) \qquad (2.45)_1$$

$$u_i(0,\Omega) = a_i \quad u'_i(0,\Omega) = b_i \qquad (2.45)_2$$

where (a_1, b_1) and (a_2, b_2) are two pairs of arbitrary constants satisfying the condition

$$a_1b_2 - a_2b_1 \neq 0 \qquad (2.46)$$

Proof: We form the Wronskians $W_f[u_1, U_{f1}]$ and $W_f[u_2, U_{f1}]$, and then use (2.34), which leads to

$$W_f[u_1,U_{f1}] = u_1(0,\Omega)U'_{f1}(0,\Omega) - u'_1(0,\Omega)U_{f1}(0,\Omega) = u_1(1,\Omega)U'_{f1}(1,\Omega) - u'_1(1,\Omega)U_{f1}(1,\Omega) \qquad (2.47)_1$$

$$W_f[u_2,U_{f1}] = u_2(0,\Omega)U'_{f1}(0,\Omega) - u'_2(0,\Omega)U_{f1}(0,\Omega) = u_2(1,\Omega)U'_{f1}(1,\Omega) - u'_2(1,\Omega)U_{f1}(1,\Omega) \qquad (2.47)_2$$

Invoking the boundary conditions $(2.11)_{1\text{-}2}$ and initial conditions $(2.45)_2$, $(2.47)_{1\text{-}2}$ reduces to

$$a_1U'_{f1}(0,\Omega) - u_1(1,\Omega)U'_{f1}(1,\Omega) = b_1 \qquad (2.48)_1$$

$$a_2U'_{f1}(0,\Omega) - u_2(1,\Omega)U'_{f1}(1,\Omega) = b_2 \qquad (2.48)_2$$

If Ω is not an eigenvalue of (2.9), we can select the constants a_1 and a_2 such that

$$a_1u_2(1,\Omega) - a_2u_1(1,\Omega) \neq 0 \qquad (2.49)$$

Therefore, the algebraic equations $(2.48)_{1\text{-}2}$ have the unique solution given by $(2.44)_{1\text{-}2}$. From $(2.44)_{1\text{-}2}$ we conclude that the problem of finding the derivatives $U'_{f1}(0, \Omega)$ and $U'_{f1}(1, \Omega)$, which are needed to specify the dispersion relationship (2.27), reduces to solving the initial value problem $(2.45)_{1\text{-}2}$ to obtain the values of u_i at the end $x = 1$. An identical procedure is followed to find the derivatives $U'_{m1}(0, \Omega)$ and $U'_{m1}(1, \Omega)$.

In order to calculate stresses in the entire interval (0, 1), we need to find the derivatives of the fundamental solutions $U_{mi}(x, \Omega)$ and $U_{fi}(x, \Omega)$. They can be calculated by the following procedure.

Proposition 4. If Ω is not an eigenvalue of $(2.9)_{1\text{-}2}$ we can use Chung's method to express the derivatives of the first and second fundamental solutions $U_{f1}(x, \Omega)$ and $U_{f2}(x, \Omega)$ as

$$U'_{f1}(x,\Omega) = \sqrt{\frac{V_f(x,\Omega)}{V_f(0,\Omega)}}\,(U'_{f1}(0,\Omega)L_{f0}(x,\Omega)$$

$$+ U'_{f1}(1,\Omega)L_{f1}(x,\Omega)) \qquad (2.50)_1$$

$$U'_{f2}(x,\Omega) = -\sqrt{\frac{V_f(x,\Omega)}{V_f(0,\Omega)}}\,(U'_{f1}(1,\Omega)L_{f0}(x,\Omega) + U'_{f1}(0,\Omega)L_{f1}(x,\Omega)) \qquad (2.50)_2$$

and likewise for $U_{m1}(x, \Omega)$ and $U_{m2}(x, \Omega)$

$$U'_{m1}(x,\Omega) = \sqrt{\frac{V_m(x,\Omega)}{V_m(0,\Omega)}}\,(U'_{m1}(0,\Omega)L_{m0}(x,\Omega) + U'_{m1}(1,\Omega)L_{m1}(x,\Omega)) \qquad (2.51)_1$$

$$U'_{m2}(x,\Omega) = -\sqrt{\frac{V_m(x,\Omega)}{V_m(0,\Omega)}}\,(U'_{m1}(1,\Omega)L_{m0}(x,\Omega) + U'_{m1}(0,\Omega)L_{m1}(x,\Omega)) \qquad (2.51)_2$$

where

$$L_{m0}(x,\Omega) = \int_{[B_\tau=0]} \exp\left(\int_0^\tau H_m(B_t,\Omega)dt\right)dp^x \qquad (2.52)_1$$

$$L_{m1}(x,\Omega) = \int_{[B_\tau=1]} \exp\left(\int_0^\tau H_m(B_t,\Omega)dt\right)dp^x \qquad (2.52)_2$$

$$L_{f0}(x,\Omega) = \int_{[B_\tau=0]} \exp\left(\int_0^\tau H_f(B_t,\Omega)dt\right)dp^x \qquad (2.53)_1$$

$$L_{f1}(x,\Omega) = \int_{[B_\tau=1]} \exp\left(\int_0^\tau H_f(B_t,\Omega)dt\right)dp^x \qquad (2.53)_2$$

The functions H_m and H_f are given by

$$H_m(x,\Omega) = V_m(x,\Omega) + \frac{1}{2}\frac{d^2}{dx^2}\left[\log V_m(x,\Omega)\right] - \frac{1}{4}\frac{d}{dx}\left[\log V_m(x,\Omega)\right] \qquad (2.54)_1$$

$$H_f(x,\Omega) = V_f(x,\Omega) + \frac{1}{2}\frac{d^2}{dx^2}\left[\log V_f(x,\Omega)\right] - \frac{1}{4}\frac{d}{dx}\left[\log V_f(x,\Omega)\right] \qquad (2.54)_2$$

$U'_{f1}(1, \Omega)$, $U'_{f1}(1, \Omega)$, $U'_{f2}(0, \Omega)$, and $U'_{f2}(1, \Omega)$ are known after considering (2.31) and (2.44).

Proof: We introduce a new function $Y_1(x, \Omega)$ by

$$Y_1(x,\Omega) = \sqrt{\frac{V_f(0,\Omega)}{V_f(x,\Omega)}}\,U'_{f1}(x,\Omega) \qquad (2.55)$$

Substituting (2.55) into $(2.9)_2$ and using the symmetry condition $V_f(0, \Omega) = V_f(1, \Omega)$, we obtain a boundary value problem for the unknown function $Y(x, \Omega)$ as follows

$$\frac{d^2 Y_1}{dx^2} + H_f(x,\Omega)Y_1(x,\Omega) = 0 \qquad (2.56)_1$$

$$Y_1(0,\Omega) = U'_{f1}(0,\Omega), \quad Y_1(1,\Omega) = U'_{f1}(1,\Omega) \qquad (2.56)_2$$

Thus, following the same procedure that lead to $(2.12)_{1-2}$, we obtain the first and second fundamental solutions of the above boundary value problem as expressed in $(2.50)_{1-2}$ and $(2.51)_{1-2}$.

3 Pass and Stop Bands in Nonhomogeneous Composites

The most important feature regarding wave propagation in a periodically layered, elastic, homogeneous medium is the presence of stop band characteristics. Next we investigate how this characteristic is affected by specific material nonhomogeneities.

When the basic dispersion relationship (2.36) is expanded, the following two equations emerge

$$e^{-2k_2d}\cos 2k_2d + F(\Omega)e^{-k_2d}\cos k_1d + 1 = 0 \qquad (3.1)_1$$

$$e^{-k_2d}\sin k_1d\left[e^{-k_2d}\cos k_1d + \frac{1}{2}F(\Omega)\right] = 0 \qquad (3.1)_2$$

where the complex Floquet wave number k has been decomposed into a real part k_1 and an imaginary part k_2, called the dispersion coefficient and dissipation coefficient, respectively. In order to find the specific dependence of k_1 and k_2 on Ω, $(3.1)_{1-2}$ possesses must be solved simultaneously. The solution depends on the magnitude of the function $F(\Omega)$ as follows

(1) When

$$\left|\frac{1}{2}F(\Omega)\right| \leq 1 \qquad (3.2)$$

$(3.1)_{1-2}$ possesses the unique solution

$$\cos k_1d = -\frac{1}{2}F(\Omega) \qquad (3.3)_1$$

$$k_2d = 0 \qquad (3.3)_2$$

Thus, the dissipation coefficient k_2 vanishes, and the pass band in the dispersion spectrum consists of all nondimensional frequencies Ω which satisfy (3.2). In other words, harmonic waves are propagated without attentuation for values of Ω which satisfy (3.2).

(2) When

$$\left|\frac{1}{2}F(\Omega)\right| > 1 \qquad (3.4)$$

then $(3.1)_{1-2}$ possess the solution

$$k_1d = n\pi \qquad (n = 0, 1, 2, \ldots) \qquad (3.5)_1$$

$$k_2d = -\log\left(\left|\frac{F(\Omega)}{2}\right| - \sqrt{\left(\frac{F(\Omega)}{2}\right)^2 - 1}\right) \qquad (3.5)_2$$

Here the dissipation coefficient k_2 does not vanish. Therefore, when the frequency Ω results in (3.4) being satisfied, harmonic waves are attenuated as they pass through the medium. This is the proof of the presence of stop bands, and $(3.5)_2$ predicts how the dissipation coefficient depends on the frequency.

We have assumed that the nondimensional frequency Ω is not an eigenvalue of $(2.9)_{1-2}$, so that the function $F(\Omega)$ must be finite. We now examine the case when Ω becomes an eigenvalue of $(2.9)_1$ or $(2.9)_2$. In the following, we refer to the eigenvalue as Ω^*.

Proposition 5. $U'_{f1}(1, \Omega)$ and $U'_{f2}(1, \Omega)$ tend to infinity if and only if Ω tends to Ω^*.

Proof: Let $u^* = u^*(x, \Omega^*)$ be one of the eigenfunctions corresponding to Ω^*. According to the definition of the eigenvalue, we must have

$$u^*(0,\Omega^*) = u^*(1,\Omega) = 0 \qquad (3.6)$$

If we further assume that

$$u_1 = u^*(x,\Omega^*) \quad a_1 = 0 \quad b_1 = u^{*\prime}(0,\Omega^*) \neq 0 \qquad (3.7)$$

From $(2.44)_{1-2}$, we find that when $a_1 = 0$ and $u_1(1, \Omega) = 0$

$$U'_{f1}(0,\Omega) \to \infty \quad U'_{f1}(1,\Omega) \to \infty \qquad (3.8)$$

On the other hand, for $|u'_1(0, \Omega^*)|$ to become unbounded, it is seen from $(2.44)_2$ that for any two nontrivial constants $u_2(0, \Omega)$ and $u_2(1, \Omega)$, the expression

$$u_1^{*\prime}(1,\Omega^*)u_2(0,\Omega) - u_1^*(0,\Omega^*)u_2'(0,\Omega) \to 0 \qquad (3.9)$$

Obviously, this is the case only if both $u_1^*(0, \Omega^*)$ and $u_1^*(1, \Omega^*)$ tend to zero simultaneously. This also means that Ω must be one of the eigenvalues of $(2.9)_{1-2}$.

The eigenfrequency Ω^* could reside in either the pass band or the stop band. This depends on the behavior of function $F(\Omega)$ as Ω tends to Ω^*. We take

$$F^* = \lim_{\Omega \to \Omega^*} F(\Omega) \tag{3.10}$$

Therefore, as stated in (3.2), if $|F^*| \le 2$, Ω^* is in the pass band, otherwise, it falls in a stop band. For nonhomogeneous composites, there may exist a special eigenfrequency where $|F^*|$ becomes infinite. We would call such an eigenfrequency a *pole* in the frequency spectrum. If a pole would occur, the dissipation coefficient $k_2(\Omega^*)$ would become infinite. The amplitude of any harmonic waves would be immediately attenuated at this frequency. In the next section, we show by example that at this special frequency all eigenfrequencies for homogeneous layered composites lie entirely in the pass band. Therefore, a pole cannot exist for homogeneous composites. This raises a very natural question: For nonhomogeneous composites, can a pole actually occur? It has been proved that the derivatives of the first and second fundamental solutions $U_{f1}(x, \Omega)$ and $U_{f2}(x, \Omega)$ become infinite at both ends $x = 0$ and $x = 1$ when Ω is one of the eigenfrequencies. In this case, the spectrum function $F(\Omega)$ is approximated by

$$F(\Omega) \approx \frac{(U'_{f1}(1,\Omega))^2 - (U'_{f1}(0,\Omega))^2}{U'_{f1}(1,\Omega)} + \text{const}\ \frac{U'_{f1}(0,\Omega)}{U'_{f1}(1,\Omega)} \tag{3.11}$$

Thus, if $|U'_{f1}(0, \Omega)| \neq |U'_{f1}(1, \Omega)|$ as Ω tends to Ω^*, then $F^*(\Omega^*)$ tends to infinity. This must indicate the presence of a "pole." A more detailed discussion on this interesting problem is actually needed and will be addressed in another paper. In what follows, we give some examples to illustrate the differences in the behavior of the dispersion relationship between homogeneous and nonhomogeneous composites.

4 Examples and Discussion of Numerical Results

Example 1. As a special case of the general theory presented above, we will calculate the spectrum function and associated dispersion characteristics for an elastic composite with homogeneous layers. In this case, the mass density and elastic moduli are constant within each matrix and fiber layer. We then have

$$V_m = \pi^{*2}\Omega^2, \quad V_f = \pi^2\Omega^2, \tag*{(4.1)$_{1-2}$}$$

The first and second fundamental solutions of $(2.7)_{1-2}$ are given by

$$U_{m1}(x,\Omega) = \frac{\sin \pi^*\Omega(1-x)}{\sin \pi^*\Omega} \quad U_{m2}(x,\Omega) = \frac{\sin \pi^*\Omega x}{\sin \pi^*\Omega} \tag*{(4.2)$_{1-2}$}$$

$$U_{f1}(x,\Omega) = \frac{\sin \pi\Omega(1-x)}{\sin \pi\Omega} \quad U_{f2}(x,\Omega) = \frac{\sin \pi\Omega x}{\sin \pi\Omega} \tag*{(4.3)$_{1-2}$}$$

Therefore, the derivatives of the first two fundamental solutions $U_{m1}(x, \Omega)$ and $U_{f1}(x, \Omega)$ are given by

$$U'_{m1}(x,\Omega) = -\pi^*\Omega\ \frac{\cos \pi^*\Omega(1-x)}{\sin \pi^*\Omega} \tag*{(4.4)$_1$}$$

$$U'_{f1}(x,\Omega) = -\pi\Omega\ \frac{\cos \pi\Omega(1-x)}{\sin \pi\Omega} \tag*{(4.4)$_2$}$$

From $(4.2)_{1-2}$, $(4.3)_{1-2}$, and $(4.4)_{1-2}$,

$$U'_{m1}(0,\Omega) = -\pi^*\Omega \cot \pi^*\Omega,$$

$$U'_{m1}(1,\Omega) = -\pi^*\Omega \csc \pi^*\Omega \tag*{(4.5)$_{1-2}$}$$

$$U'_{f1}(0,\Omega) = -\pi\Omega \cot \pi\Omega \quad U'_{f1}(1,\Omega) = -\pi\Omega \csc \pi\Omega \tag*{(4.6)$_{1-2}$}$$

Substituting $(4.5)_{1-2}$ and $(4.6)_{1-2}$ into (2.37), we find the spectrum function has the form

$$G(\Omega) = -\frac{F(\Omega)}{2} = \cos \pi\Omega \cos \pi^*\Omega - \lambda \sin \pi\Omega \sin \pi^*\Omega \tag{4.7}$$

where $\lambda = \frac{1}{2}\left[\frac{\rho_{mo}c_{mo}}{\rho_{fo}c_{fo}} + \frac{\rho_{fo}c_{fo}}{\rho_{mo}c_{mo}}\right]$.

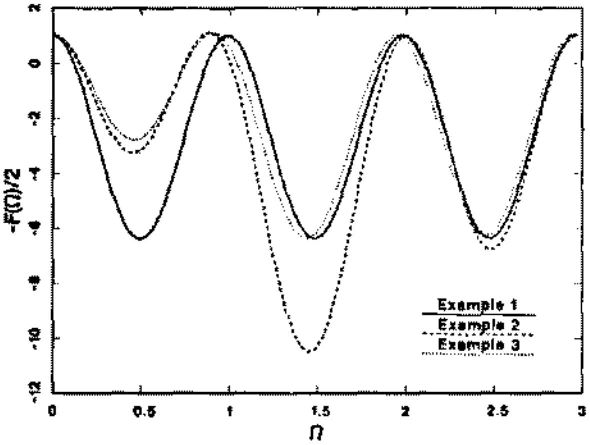

Fig. 2 Spectrum function for three example problems

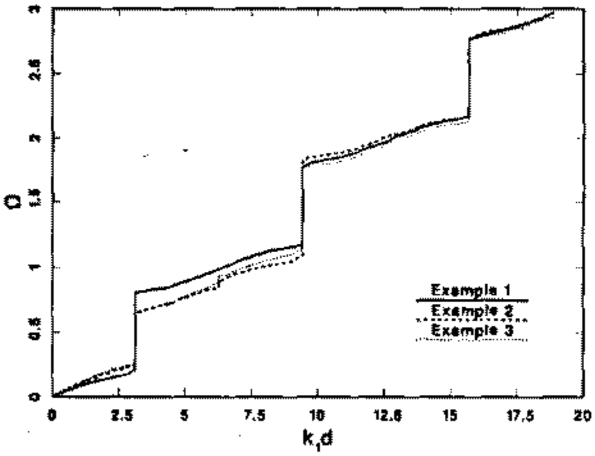

Fig. 3 Dispersion coefficient for three example problems

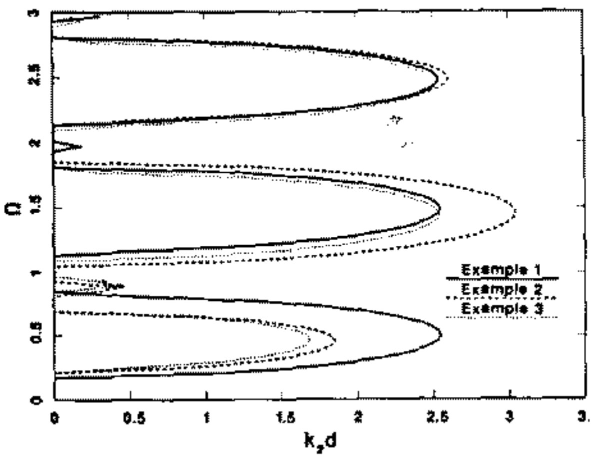

Fig. 4 Dissipation coefficient for three example problems

Thus, based on previous arguments, we obtain the spectral behavior as follows

$$\cos k_1 d = \begin{cases} G(\Omega), & \text{for } |G(\Omega)| \le 1 \\ (-1)^n, & \text{for } |G(\Omega)| > 1 \end{cases} \tag{4.8}$$

where n is a positive integer and

$$k_2 d = \begin{cases} 0, & \text{for } |G(\Omega)| \le 1 \\ \log\left(|G(\Omega)| - \sqrt{(G(\Omega))^2 - 1}\right), & \text{for } |G(\Omega)| > 1 \end{cases} \tag{4.9}$$

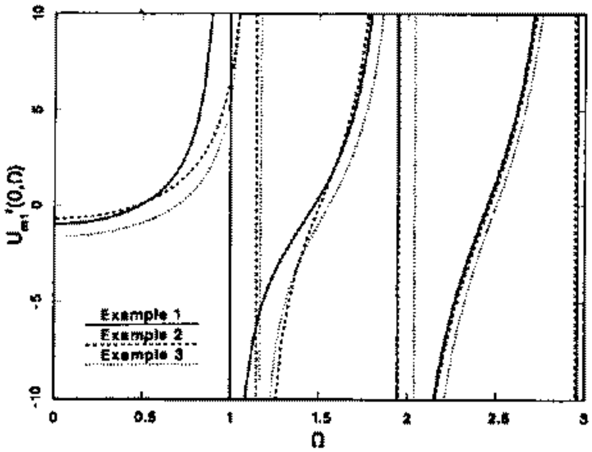

Fig. 5 Derivative of fundamental solution U_{m1} at $x = 0$ for three example problems

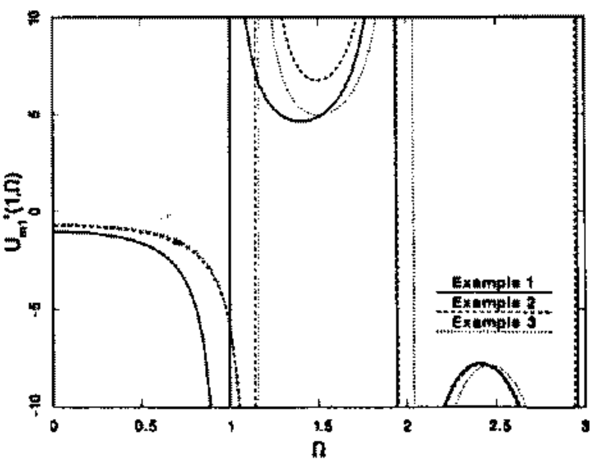

Fig. 6 Derivative of fundamental solution U_{m1} at $x = 1$ for three example problems

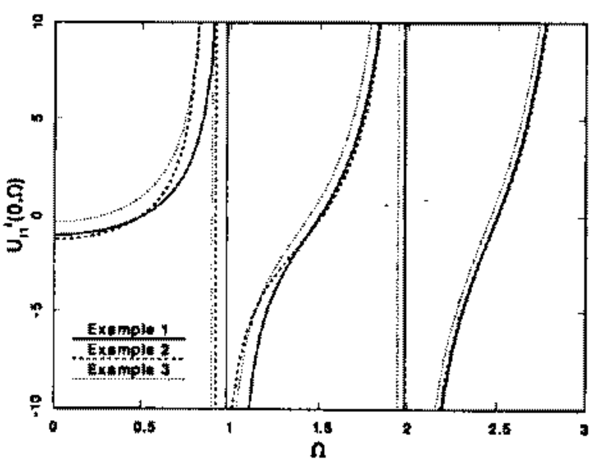

Fig. 7 Derivative of fundamental solution U_{f1} at $x = 0$ for three example problems

Thus, when the relation $|G(\Omega)| \leq 1$ is satisfied, the corresponding frequency is in the pass band. Otherwise, the frequencies are in the stop band. These results are the same as given by Lee and Yang [5] and Delph et al [7–8]. From these solutions, we see that there are two sets of eigenfrequencies

$$\Omega_m^* = \frac{\pi}{\pi^*} n, \quad \Omega_f^* = n \qquad (4.10)_{1-2}$$

at which both derivatives $U_{f1}'(x, \Omega)$, and $U_{m1}'(x, \Omega)$ tend to infinity at the boundaries $x = 0$ and $x = 1$. After substituting $(4.10)_{1-2}$ into (4.7), we obtain

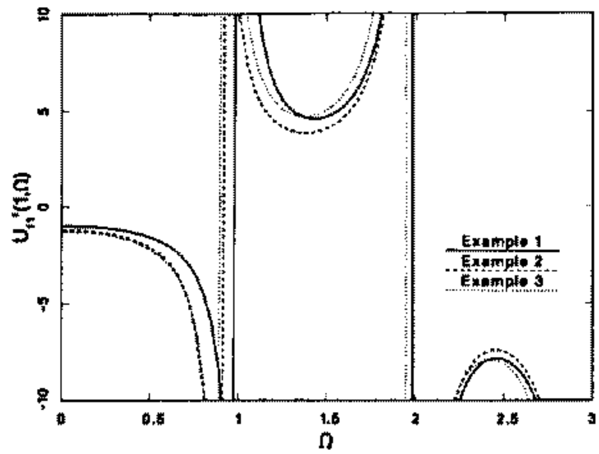

Fig. 8 Derivative of fundamental solution U_{f1} at $x = 1$ for three example problems

$$G(\Omega_f^*) = (-1)^n \cos n\pi^* \qquad (4.11)_1$$

$$G(\Omega_m^*) = (-1)^n \cos \frac{\pi^2}{\pi^*} n \qquad (4.11)_2$$

For these eigenfrequencies, we always have

$$|G(\Omega_f^*)| \leq 1, \quad |G(\Omega_m^*)| \leq 1 \qquad (4.12)_{1-2}$$

which means that for homogeneous, layered materials, all eigenfrequencies are in the pass band.

Figure 2 shows the spectrum function $G(\Omega) = -\frac{1}{2}F(\Omega)$ versus Ω for this composite. In this example we have assumed that $h_m/h_f = 0.25$, $D_m(0)/D_f(0) = 0.02$, and $\rho_m(0)/\rho_f(0) = 0.33$. These values are the same as those used in the paper by Delph et al. [7]. Figures 3 and 4 show the dispersion relations Ω versus k_1d and Ω versus k_2d, respectively. Figures 5–8 show the behavior of the derivatives of fundamental solutions $U_{m1}'(0, \Omega)$, $U_{m1}'(1, \Omega)$, $U_{f1}'(0, \Omega)$, and $U_{f1}'(1, \Omega)$, respectively.

Example 2. As an example of a nonhomogeneous elastic composite, we will consider the following variation of the material constants in each of the layers

$$\rho_m(x) = 1 - \frac{\rho_o}{2}[1 + \cos \pi(2x-1)] \qquad (4.13)_1$$

$$D_m(x) = 1 - \frac{D_o}{2}[1 + \cos \pi(2x-1)] \qquad (4.13)_2$$

and

$$\rho_f(x) = 1 + \frac{\rho_o}{2}[1 + \cos \pi(2x-1)] \qquad (4.14)_1$$

$$D_f(x) = 1 + \frac{D_o}{2}[1 + \cos \pi(2x-1)] \qquad (4.14)_2$$

where ρ_o and D_o are two positive parameters each less than unity.

The spectrum function $-F(\Omega)/2$ is shown in Fig. 2 for this nonhomogeneous composite. We have assumed the same values for the parameters h_m/h_f, $D_m(0)/D_f(0)$, and $\rho_m(0)/\rho_f(0)$ as in Example 1. The parameters D_o and ρ_o were both chosen to be 0.5. Figures 3 and 4 show the dispersion relations, while Figs. 5–8 show the derivatives of the fundamental solutions for this example.

Example 3. As another example of a nonhomogeneous elastic composite, we will assume the following quadratic variation of the material constants

$$\rho_m(x) = 1 + \rho_o x(1-x) \qquad (4.15)_1$$

$$D_m(x) = 1 + D_o x(1-x) \qquad (4.15)_2$$

and

$$\rho_f(x) = 1 + \rho_o x(1-x) \qquad (4.16)_1$$

$$D_f(x) = 1 + D_o x(1-x) \qquad (4.16)_2$$

where ρ_o and D_o are two positive parameters each less than 4.0. For this case, the values of all parameters were taken to be the same as in Example 2, except D_o and ρ_o were both equal to 2.0. This meant that at $x = 0.5$, the magnitude of the material properties was the same as in Example 2. The spectrum function, dispersion relations and the derivatives of the fundamental solutions are shown in Fig. 2, Figs. 3 and 4, and Figs. 5–8, respectively.

Figure 2 exemplifies the effect of material nonhomogeneity on the behavior of the spectrum function. The nonhomogeneity is seen to affect both the amplitude and phase of the spectrum function. Figures 3 and 4 show how the material nonhomogeneity changes the basic dispersion relationship. The vertical line segments in Fig. 3 are the stop bands and indicate the range of frequency where attenuation of the wave amplitude will occur. The nonhomogeneity is seen to affect the width of the stop bands, particularly at low frequencies. A sharp decrease of the attenuation coefficient $k_2 d$ at low frequencies can be seen from Fig. 4. The material nonhomogeneities which have been considered are seen not to affect the high frequency behavior of the spectrum function and the corresponding dispersion relationship when compared to composites constructed of homogeneous layers.

Figures 5 and 6 show the derivatives of the fundamental solutions U'_{m1} $(0, \Omega)$ and U'_{m1} $(1, \Omega)$. Eigenfrequencies are indicated where the derivatives become unbounded. Figures 7 and 8 show the derivatives U'_{f1} $(0, \Omega)$ and U'_{f1} $(1, \Omega)$. Here eigenfrequencies also exist. On comparing these results with Figs. 2 and 3, it is seen that the eigenfrequencies lie within the stop bands for the nonhomogeneous composites and within the pass bands for the homogeneous composites. In addition, a "pole" was not discovered during the calculation. Further studies to either discover or rule out the presence of such a feature is necessary.

Our attention has focused only on the investigation of the dispersion relations for nonhomogeneous composites by combining Floquet's wave theory with Wronskian properties of the fundamental solutions of the associated differential equations. We will defer the calculation of the vibrational mode shapes and discussion of further details of Chung's probability theory to a later paper.

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