

## Fracture of nonhomogeneous materials

J.W. EISCHEN

*Department of Mechanical and Aerospace Engineering, Campus Box 7910, North Carolina State University, Raleigh, NC 27695, USA*

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**Abstract.** The nonhomogeneous materials considered in this work are of a class whose elastic moduli are specified by continuous and generally differentiable functions of the spatial coordinates. The elastic stress and displacement fields near a crack tip in a two-dimensional nonhomogeneous cracked body are derived utilizing an extension of the Williams eigenfunction expansion technique. The nature of the stress and strain singularity is ascertained to be precisely of the same form as the well-known inverse square root stress singularity near a crack tip in a homogeneous material, independent of the functional form of the elastic moduli variation. A new quasi-path-independent integral has been generated which proves useful for computing the energy release rate and mixed-mode stress intensity factors in nonhomogeneous cracked bodies. The integral is used in conjunction with finite element analysis for purposes of computing stress intensity factors. Numerical results are compared with certain exact solutions which are available for nonhomogeneous cracked bodies. Cracked composite bodies have traditionally been modeled and analyzed as possessing discontinuous elastic moduli, but are treated here as having rapid, but smooth variations of the material properties.

### 1. Introduction and background

Current engineering designs frequently involve situations where nonhomogeneous materials are either present naturally, or are used intentionally to attain a required structural performance. The elastic moduli depend on position in a nonhomogeneous material. Generally, two classes of such nonhomogeneous materials are discussed. The first class features an abrupt discontinuity in the magnitude of the material constants, such as in laminated composite structures. The second class involves materials where the elastic moduli vary smoothly. Soils, foundations, and geologic structures are all examples of where such a variation occurs naturally. When intentional bonding of two materials creates a smooth, but rapid variation in the elastic moduli near an interface, another example of the second class occurs. Issues relevant to fracture analysis of this latter class of materials form the main impetus of this paper.

Many researchers have been concerned with developing solutions for the states of stress and strain near crack tips located in composite media. For analytical purposes, composites are normally idealized as structures possessing elastic moduli which are only piecewise constant. In reality, abrupt discontinuities in the elastic moduli cannot exist because the interface between solids of differing elastic constants must be diffuse. For example, when two such solids are bonded, the material in the proximity of the bond possesses continuous, rapidly varying elastic moduli. The elastic fields around a crack tip which resides in a material whose elastic moduli are continuous functions of position must be understood in order to predict the fracture susceptibility of such materials. The elastic crack tip fields in such materials have not been studied nearly as vigorously as problems which have been

modeled, at least mathematically, as containing abrupt modulus changes. Interest in media with smoothly varying moduli have arisen partly because of possible applications to soil mechanics and geophysics. It is known that the rigidity of natural soils increases with distance from the surface of the earth (see Kassir [1]), and therefore the analysis of crack-like features necessitates an understanding of the fracture of nonhomogeneous materials. Fuse-bonded materials used in the electronics industry also exhibit characteristics of the type being discussed.

The work reported in this paper has been directed towards finite element computation of fracture parameters (stress intensity factors) in cracked bodies of arbitrary geometrical configuration whose elastic moduli are "smooth" functions of the spatial coordinates. Successful numerical work pertaining to fracture mechanics hinges on a fundamental understanding of the elastic crack tip fields. As a precursor to the numerical work, several theoretical issues were settled. In a recent paper by Delale [2], the following statement was made: "Even though no systematic study of the problem appears to have been made, it is reasonable to expect that in nonhomogeneous materials with continuous and generally differentiable elastic moduli the nature of the stress singularity at a crack tip would be identical to that of a homogeneous solid". And, Erdogan [3] stated "... if the crack is embedded into a nonhomogeneous medium with smoothly varying elastic properties the square root nature of the stress singularity *seems* to remain unchanged". It will be shown that regardless of the functional form of the modulus variation, an  $r^{-1/2}$  stress and strain singularity exists at the crack tip. The angular variation of the singular stress field and the associated displacements around a crack tip in a nonhomogeneous material are shown to be exactly the same as the angular variation in a homogeneous material.

The literature concerned with analytical determination of stress intensity factors in nonhomogeneous materials whose elastic moduli vary smoothly is quite sparse, particularly for mode *I* and mode *II* deformation states. All solutions to date have addressed fracture of two-dimensional elastic media of infinite extent. The mode *I* stress intensity factor for an incompressible elastic body with a shear modulus which varies in the direction perpendicular to the crack line according to  $\mu = \bar{\mu}(1 + \beta|y|)$  was studied by Rogers [4]. A singular integral equation was obtained which did not admit a closed-form solution. A variation of this problem was investigated by Gerasoulis [5]. The incompressibility assumption was relaxed and the shear modulus was taken to vary inversely as a function of distance from the crack line according to  $\mu = \bar{\mu}/(1 + \beta|y|)$ . The work by Delale [2] was the first to deal with the relatively more complicated problem of mode *I* deformation in the presence of a modulus variation in the direction parallel to the crack line. Young's modulus was given by  $E = \bar{E} \exp(\beta x)$ , while Poisson's ratio was constant. A singular integral equation was derived and then solved numerically to produce the stress intensity factor and crack surface displacement for a variety of load cases. Fabrikant [6] has analyzed a semi-infinite crack in a three-dimensional nonhomogeneous elastic body whose shear modulus varies according to a power law in the coordinate direction normal to the crack plane. Solutions for mode *II* cracks in nonhomogeneous materials were seemingly absent in the literature. Dhaliwal [4], Delale [8], and Erdogan [9] have analyzed crack problems in nonhomogeneous materials under mode *III* deformations. The analysis by Erdogan [9] is particularly interesting in that a problem where a shear modulus possessing a discontinuous derivative is studied. Furthermore, an analytical solution has not been obtained for the case of a continuously variable Poisson's ratio. The only prior direct numerical (as opposed to integral equations) work

related to fracture of nonhomogeneous materials of the type under discussion was by Atkinson [10]. A boundary integral equation crack tip analysis was executed to compute stress intensity factors in finite nonhomogeneous bodies subjected to mode *III* deformation.

Common features (or limitations) of all the prior analytical solutions for cracks in nonhomogeneous materials were the presence of an infinite domain and relatively simple functional forms for the modulus variation. In all cases the modulus variation was selected to render the governing partial differential equations mathematically tractable. The modulus variation is typically taken as a simple exponential or power-law along one of the Cartesian coordinate directions. Use of the finite element method is shown in this paper to overcome the limitations mentioned above. Realistic problems involving cracks require consideration of finite size effects. The finite element method can routinely handle difficult physical topologies. Secondly, the general procedures to be presented in the subsequent development do not restrict the functional form of the elastic modulus variation, including a variation in Poisson's ratio.

The organization of the paper is as follows. Section 2 contains a derivation of the stress and displacement fields near crack tips in arbitrary two-dimensional nonhomogeneous elastic bodies. Section 3 sets forth the necessary theoretical basis for determining stress intensity factors in nonhomogeneous cracked bodies via domain-independent integrals. An extension of the standard *J* integral (Rice [11]), which requires that a domain integral be appended to the usual line integral term, is developed. Simple relationships are shown to exist between the new *J\** vector and the mode *I* and *II* stress intensity factors. Section 4 contains several numerical examples which verify the theoretical and computational issues described herein.

## 2. Stress analysis of cracks in nonhomogeneous elastic solids

A direct way to ascertain the nature of the near-tip fields in a cracked body is the eigenfunction expansion technique due to Williams [12]. This procedure has been widely exploited in fracture mechanics analysis of homogeneous bodies in the past. An extension of this conventional procedure is used to establish the general form of the stress and displacement fields near a crack tip in a nonhomogeneous material. Figure 1 shows a crack embedded in a two-dimensional nonhomogeneous elastic body. Cartesian and cylindrical coordinate systems have been fixed at the crack tip. Applied tractions and/or specified displacements on the boundary of the body are shown schematically, and are assumed to result in a state of generalized plane stress or plane strain. Body forces are neglected, and crack faces are assumed traction-free.

The stress equilibrium equations are satisfied identically by an Airy stress function  $\phi(r, \theta)$  such that

$$\sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) \quad (1)$$

where  $\sigma_{rr}$ ,  $\sigma_{\theta\theta}$ , and  $\sigma_{r\theta}$  are the cylindrical components of the stress tensor. The strain compatibility equation expressed in terms of cylindrical coordinates is

$$\frac{1}{r^2} \frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} + \frac{\partial^2 \varepsilon_{\theta\theta}}{\partial r^2} - \frac{1}{r} \frac{\partial \varepsilon_{rr}}{\partial r} + \frac{2}{r} \frac{\partial \varepsilon_{\theta\theta}}{\partial r} = \frac{2}{r} \frac{\partial^2 \varepsilon_{r\theta}}{\partial r \partial \theta} + \frac{2}{r^2} \frac{\partial \varepsilon_{r\theta}}{\partial \theta} \quad (2)$$

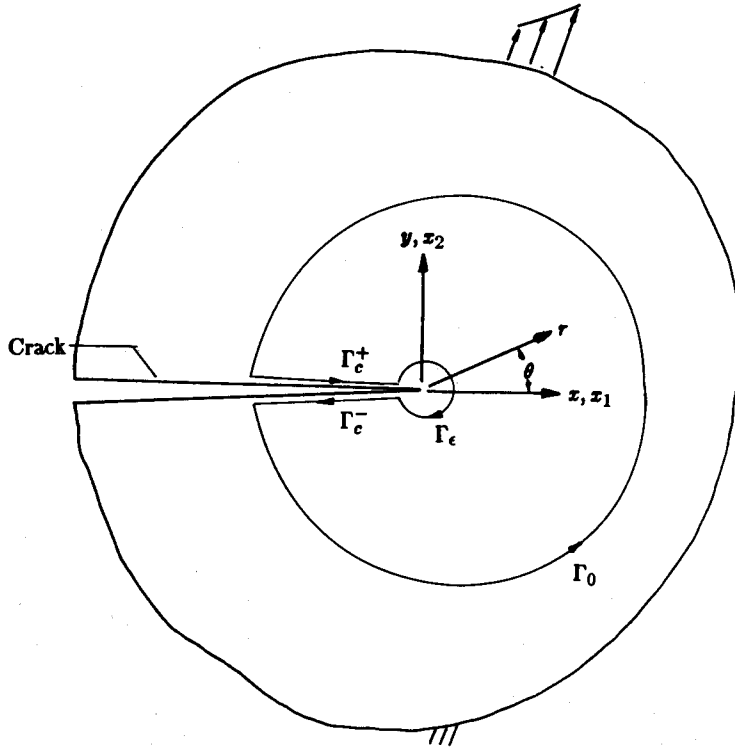


Fig. 1. Schematic of cracked body.

The elastic moduli  $E$  and  $\nu$  are assumed to vary with position according to

$$E = E(r, \theta), \quad \nu = \nu(r, \theta) \quad (3)$$

where  $E(r, \theta)$  and  $\nu(r, \theta)$  are continuous, bounded, and generally differentiable functions. Furthermore, it is required that  $E > 0$  and  $-1 < \nu < \frac{1}{2}$  everywhere in the domain. When the stress-strain relations and (1) are substituted into (2), the following equation governing the stress function  $\phi$  for generalized plane stress conditions is obtained

$$\begin{aligned} \nabla^4 \phi + & \left[ \frac{2}{E^2} \left( \frac{\partial E}{\partial r} \right)^2 - \frac{1}{E} \frac{\partial^2 E}{\partial r^2} \right] \left[ \frac{\partial^2 \phi}{\partial r^2} - \nu \left( \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) \right] \\ & + \left[ \frac{2}{E^2} \left( \frac{\partial E}{\partial \theta} \right)^2 - \frac{1}{E} \frac{\partial^2 E}{\partial \theta^2} \right] \left[ \frac{1}{r^3} \frac{\partial \phi}{\partial r} + \frac{1}{r^4} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{\nu}{r^2} \frac{\partial^2 \phi}{\partial r^2} \right] \\ & + \frac{1}{E} \frac{\partial E}{\partial r} \left[ -2 \frac{\partial^3 \phi}{\partial r^3} - \frac{2}{r} \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^3} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial \phi}{\partial r} + \nu \left( \frac{2}{r^2} \frac{\partial^3 \phi}{\partial r \partial \theta^2} + \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} - \frac{2}{r^3} \frac{\partial^2 \phi}{\partial \theta^2} \right) \right] \\ & + \frac{1}{E} \frac{\partial E}{\partial \theta} \left[ \frac{2\nu}{r^2} \frac{\partial^3 \phi}{\partial r^2 \partial \theta} - \frac{2}{r^4} \frac{\partial^3 \phi}{\partial \theta^3} - \frac{2}{r^3} \frac{\partial^2 \phi}{\partial r \partial \theta} \right] \end{aligned}$$

$$\begin{aligned}
& + \left[ 2 \frac{(1 + \nu)}{E^2} \frac{\partial E}{\partial r} \frac{\partial E}{\partial \theta} - \frac{(1 + \nu)}{E} \frac{\partial^2 E}{\partial r \partial \theta} \right] \left[ \frac{2}{r^2} \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{2}{r^3} \frac{\partial \phi}{\partial \theta} \right] \\
& + \frac{(1 + \nu)}{E} \left[ \frac{\partial E}{\partial r} \left( -\frac{2}{r^2} \frac{\partial^3 \phi}{\partial r \partial \theta^2} + \frac{2}{r^3} \frac{\partial^2 \phi}{\partial \theta^2} \right) + \frac{\partial E}{\partial \theta} \left( -\frac{2}{r^2} \frac{\partial^3 \phi}{\partial r^2 \partial \theta} + \frac{2}{r^3} \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{2}{r^4} \frac{\partial \phi}{\partial \theta} \right) \right] \\
& - \frac{\partial^2 \nu}{\partial r^2} \left[ \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] - \frac{\partial^2 \nu}{\partial r \partial \theta} \left[ \frac{2}{r^3} \frac{\partial \phi}{\partial \theta} - \frac{2}{r^2} \frac{\partial^2 \phi}{\partial r \partial \theta} \right] - \frac{\partial^2 \nu}{\partial \theta^2} \left[ \frac{1}{r^2} \frac{\partial^2 \phi}{\partial r^2} \right] \\
& - \frac{\partial \nu}{\partial r} \left[ \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} \right] - \frac{\partial \nu}{\partial \theta} \left[ -\frac{1}{r^2} \frac{\partial^3 \phi}{\partial r^2 \partial \theta} + \frac{2}{r^3} \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{2}{r^4} \frac{\partial \phi}{\partial \theta} \right] \\
& + \frac{2}{E} \frac{\partial E}{\partial r} \frac{\partial \nu}{\partial r} \left[ \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right] + \frac{1}{E} \frac{\partial E}{\partial r} \frac{\partial \nu}{\partial \theta} \left[ -\frac{2}{r^2} \frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{2}{r^3} \frac{\partial \phi}{\partial \theta} \right] \\
& + \frac{1}{E} \frac{\partial E}{\partial \theta} \frac{\partial \nu}{\partial r} \left[ -\frac{2}{r^2} \frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{2}{r^3} \frac{\partial \phi}{\partial \theta} \right] + \frac{1}{E} \frac{\partial E}{\partial \theta} \frac{\partial \nu}{\partial \theta} \left[ \frac{1}{r^2} \frac{\partial^2 \phi}{\partial r^2} \right] \\
& = 0.
\end{aligned} \tag{4}$$

The first term in the governing equation for  $\phi$  involves the biharmonic operator just as in the case of a homogeneous material, while all remaining terms include spatial derivatives of the elastic moduli  $E$  and  $\nu$ . The corresponding relation for plane strain will not be written out here, but it is obtained by replacing  $E$  and  $\nu$  by  $E/(1 - \nu^2)$  and  $\nu/(1 - \nu)$ , respectively.

The next task is to obtain a solution for the stress function given the governing equation. A separation of variables method is applicable, with an appropriate structure of the stress function expressed as

$$\phi(r, \theta) = r^{\lambda+1} F(\theta) + r^{\lambda+2} G(\theta) + r^{\lambda+3} H(\theta) + O(r^{\lambda+4}) + \dots \tag{5}$$

where  $\lambda$  is an unspecified positive parameter. The functions  $F$ ,  $G$ , and  $H$  are unknown at this stage. The ellipses indicate an infinite sequence of terms whose form is apparent given the pattern of the first few terms in (5). Because the elastic moduli are given by continuous and differentiable functions, they can be represented by a Maclaurin series expansion about the crack tip position ( $r = 0$ ). The expansion for Young's modulus is written as

$$E(r, \theta) = E_0 \left( 1 + r E_1(\theta) + \frac{r^2}{2} E_2(\theta) + O(r^3) + \dots \right) \tag{6}$$

where  $E_0 = \text{constant}$ , and  $E_1(\theta)$ ,  $E_2(\theta)$  are smooth, bounded functions of  $\theta$ . The expressions required for  $E^{-1}$  and  $E^{-2}$  are formed in terms of  $E_0$ ,  $E_1(\theta)$ , etc., and  $E_2(\theta)$  via the binomial expansion. The Maclaurin series expansion for Poisson's ratio can be written in a similar fashion to  $E$ . In order to lessen the algebraic burden during this presentation, it will henceforth be assumed that Poisson's ratio is constant. This simplification does not alter any of the final conclusions of the present analysis. Upon substituting Eqns. (5) and (6) into (4), it

is found that the assumed form for the stress function will satisfy the governing equation if

$$\begin{aligned} & [L_{\lambda+1}^1(F)]r^{\lambda-3} + [L_{\lambda+2}^1(G) - E_1''(\theta)L_{\lambda+1}^3(F) + E_1(\theta)L_{\lambda+1}^4(F) \\ & + E_1'(\theta)L_{\lambda+1}^5(F) - (1 + \nu)(E_1'(\theta)L_{\lambda+1}^6(F) - E_1(\theta)L_{\lambda+1}^7(F) - E_1'(\theta)L_{\lambda+1}^8(F))]r^{\lambda-2} \\ & + [L_{\lambda+3}^1(H) + f(E_1(\theta), E_2(\theta), \nu, F(\theta), G(\theta))]r^{\lambda-1} + O(r^\lambda) + \dots = 0 \end{aligned} \quad (7)$$

where  $f(\dots)$  indicates a function of the arguments inside the parentheses. Terms have been grouped to multiply  $r^{\lambda-3}$ ,  $r^{\lambda-2}$ ,  $r^{\lambda-1}$ ,  $\dots$ . The symbols  $L_{\lambda+n}^m(\ )$  indicate differential operators which act on the functions  $F$ ,  $G$ , and  $H$ . The explicit form of these operators is given in the Appendix.

Equation (7) is satisfied only if each of the functions inside the brackets [ ] are set equal to zero, leading to the following system of ordinary differential equations (ODE's) which will subsequently permit a determination of  $F$ ,  $G$ , and  $H$ .

$$L_{\lambda+1}^1(F) = 0 \quad (8)$$

$$\begin{aligned} L_{\lambda+2}^1(G) &= E_1''(\theta)L_{\lambda+1}^3(F) - E_1(\theta)L_{\lambda+1}^4(F) - E_1'(\theta)L_{\lambda+1}^5(F) \\ &+ (1 + \nu)(E_1'(\theta)L_{\lambda+1}^6(F) - E_1(\theta)L_{\lambda+1}^7(F) - E_1'(\theta)L_{\lambda+1}^8(F)) \end{aligned} \quad (9)$$

$$L_{\lambda+3}^1(H) = -f(E_1(\theta), E_2(\theta), \nu, F(\theta), G(\theta)). \quad (10)$$

Equation (8) is a fourth order, linear, homogeneous ODE for the unknown function  $F(\theta)$ . Equations (9) and (10) are fourth order, linear, nonhomogeneous ODE's and must be solved accordingly for the unknown functions  $G(\theta)$  and  $H(\theta)$ , respectively. These three equations will be considered in turn. The general solution of (8) is

$$\begin{aligned} F(\theta) &= A \cos [(\lambda + 1)\theta] + B \sin [(\lambda + 1)\theta] \\ &+ C \cos [(\lambda - 1)\theta] + D \sin [(\lambda - 1)\theta] \end{aligned} \quad (11)$$

where  $A$ ,  $B$ ,  $C$ ,  $D$  are unspecified constants. The traction-free crack surface boundary conditions which must be imposed are

$$\sigma_{\theta\theta}(r, \pm\pi) = \sigma_{r\theta}(r, \pm\pi) = 0 \quad (12)$$

Imposition of these conditions leads to an eigenproblem requiring  $\lambda_n = n/2$ ,  $n = 1, 2, \dots, \infty$ . Additionally, the constants  $A$  and  $B$  are expressible in terms of  $C$  and  $D$ . The eigenfunctions  $F_n(\theta)$  are given by

$$\begin{aligned} F_n(\theta) &= C_n \left\{ \cos \left[ \left( \frac{n}{2} - 1 \right) (\theta + \pi) \right] - \cos \left[ \left( \frac{n}{2} + 1 \right) (\theta + \pi) \right] \right\} \\ &+ D_n \left\{ \sin \left[ \left( \frac{n}{2} - 1 \right) (\theta + \pi) \right] - \frac{n-2}{n+2} \sin \left[ \left( \frac{n}{2} + 1 \right) (\theta + \pi) \right] \right\}. \end{aligned} \quad (13)$$

The next task is to investigate the character of the function  $G(\theta)$ . This function must satisfy the ODE given by (9), whose general solution is comprised of the sum of a particular and complementary solution, i.e.,

$$G_n(\theta) = G_n^p(\theta) + G_n^c(\theta) \quad (14)$$

The particular solutions  $G_n^p(\theta)$  depend on the functions  $F_n(\theta)$  and the specific material nonhomogeneity, as evidenced by (9). The complementary solution is of the form

$$G_n^c(\theta) = H_n \cos\left(\frac{n}{2} + 2\right)\theta + I_n \sin\left(\frac{n}{2} + 2\right)\theta + J_n \cos\frac{n}{2}\theta + K_n \sin\frac{n}{2}\theta \quad (15)$$

where  $H_n$ ,  $I_n$ ,  $J_n$ , and  $K_n$  ( $n = 1, 2, \dots, \infty$ ) are unspecified constants. The particular solutions cannot be obtained until an explicit form of the elastic moduli variation is chosen. Even then it would not be a routine matter to determine  $G_n^p(\theta)$ . The method of variation of parameters would be an appropriate technique in this situation because the form of the homogeneous solution is known (Boyce [13]). Nevertheless, once  $G_n^p(\theta)$  is available the constants  $H_n$  thru  $K_n$  could be determined by imposing (12). A similar procedure would be followed to determine  $H(\theta)$ . Fortunately, for the intents herein, explicit knowledge of the forms of  $G(\theta)$  and  $H(\theta)$  is not necessary. This fact will become apparent in the subsequent development.

The components of stress are calculated from the following formulas, which follow from linear superposition of the eigensolutions for  $F(\theta)$ ,  $G(\theta)$ , and  $H(\theta)$

$$\begin{aligned} \sigma_{rr} = & \sum_{n=1,2,\dots}^{\infty} \left\{ r^{(n/2)-1} \left[ F_n'' + \left(\frac{n}{2} + 1\right) F_n' \right] + r^{n/2} \left[ G_n'' + \left(\frac{n}{2} + 2\right) G_n' \right] \right. \\ & \left. + r^{(n/2)+1} \left[ H_n'' + \left(\frac{n}{2} + 3\right) H_n' \right] + O(r^{(n/2)+2}) + \dots \right\} \quad (16) \end{aligned}$$

$$\begin{aligned} \sigma_{\theta\theta} = & \sum_{n=1,2,\dots}^{\infty} \left\{ r^{(n/2)-1} \left[ \frac{n}{2} \left(\frac{n}{2} + 1\right) F_n' \right] + r^{n/2} \left[ \left(\frac{n}{2} + 1\right) \left(\frac{n}{2} + 2\right) G_n' \right] \right. \\ & \left. + r^{(n/2)+1} \left[ \left(\frac{n}{2} + 2\right) \left(\frac{n}{2} + 3\right) H_n' \right] + O(r^{(n/2)+2}) + \dots \right\} \quad (17) \end{aligned}$$

$$\begin{aligned} \sigma_{r\theta} = & \sum_{n=1,2,\dots}^{\infty} \left\{ r^{(n/2)-1} \left[ -\frac{n}{2} F_n' \right] + r^{n/2} \left[ -\left(\frac{n}{2} + 1\right) G_n' \right] \right. \\ & \left. + r^{(n/2)+1} \left[ -\left(\frac{n}{2} + 2\right) H_n' \right] + O(r^{(n/2)+2}) + \dots \right\}. \quad (18) \end{aligned}$$

Only the leading term in the expressions for the components of stress, which is  $O(r^{(n/2)-1})$ , contributes to the components of stress which are singular as  $r \rightarrow 0$ . Because fracture criteria typically depend only on the character of the singular stress components (and sometimes

$O(1)$  terms), explicit determination of  $G(\theta)$  and  $H(\theta)$  is not necessary for the following reason. Since  $n$  takes on the values 1, 2, . . . , the leading terms in the expressions for the components of stress will have  $r$  raised to the powers  $-1/2, 0, 1/2, \dots$ . The second term in the expression for the components of stress ( $O(r^{n/2})$ ) will have  $r$  raised to the powers  $1/2, 1, 3/2, \dots$ . These terms are all bounded at the crack tip. Therefore, it is clear that all terms in these expressions that follow the  $O(r^{n/2})$  term result in components of stress which are non-singular. Stress components referred to a Cartesian basis ( $x, y$ ) will be given to conform with traditional presentations. Using (13) in Eqns. (16)–(18), making a coordinate transformation, and omitting terms  $O(r^{1/2})$  and above, the following approximate expressions for the stress components are obtained

$$\sigma_{xx} \cong \frac{K_I}{(2\pi r)^{1/2}} f_{xx}^I(\theta) + \frac{K_{II}}{(2\pi r)^{1/2}} f_{xx}^{II}(\theta) + \sigma_{x0} + O(r^{1/2}) + \dots \quad (19)$$

$$\sigma_{yy} \cong \frac{K_I}{(2\pi r)^{1/2}} f_{yy}^I(\theta) + \frac{K_{II}}{(2\pi r)^{1/2}} f_{yy}^{II}(\theta) + O(r^{1/2}) + \dots \quad (20)$$

$$\sigma_{xy} \cong \frac{K_I}{(2\pi r)^{1/2}} f_{xy}^I(\theta) + \frac{K_{II}}{(2\pi r)^{1/2}} f_{xy}^{II}(\theta) + O(r^{1/2}) + \dots \quad (21)$$

where the constants  $C_1$ ,  $C_2$ , and  $D_1$  have been redefined as

$$C_1 = \frac{K_{II}}{\sqrt{2\pi}}, \quad C_2 = \frac{\sigma_{x0}}{4}, \quad D_1 = \frac{-K_I}{\sqrt{2\pi}}. \quad (22)$$

The mode  $I$  and mode  $II$  stress intensity factors are designated  $K_I$  and  $K_{II}$ , respectively. The  $\sigma_{xx}$  stress component contains an  $O(1)$  term,  $\sigma_{x0}$ , called the “nonsingular stress”. This parameter has been associated with crack kinking phenomena. The ellipses designate a sequence of terms which are of successively higher degree in  $r$ . The symbols  $f_{xx}^I, f_{xx}^{II}, f_{yy}^I$ , etc. represent trigonometric functions of the angle  $\theta$  and are given by Eftis [14].

The significant results of these calculations is that the nature of the stress singularity is precisely the same as the well-known form applicable to homogeneous materials, irrespective of the particular form of the Young’s modulus variation. The terms in the series representation for the stresses proportional to  $r^{-1/2}$  and  $r^0$  are not affected by material nonhomogeneity. The degree of the singularity is preserved ( $-1/2$ ) in the presence of nonhomogeneity, as well as the angular dependence given by  $f_{xx}^I, f_{xx}^{II}$ , etc. The angular variation of the components of stress which correspond to terms  $O(r^{1/2})$  and higher ( $r^1, r^{3/2}, \dots$ ) do change due to material nonhomogeneity. The same general results are obtained if Poisson’s ratio varies “smoothly” within the domain of the cracked body. In addition to depending on the geometry and loading of the cracked body,  $K_I$  and  $K_{II}$  will depend on the variation of the elastic moduli. However, this dependence cannot be ascertained by the eigenfunction expansion procedure. A method to accomplish this task will be discussed in Section 3. Additionally, a forthcoming paper will address the problem where the elastic moduli are given by continuous functions not necessarily possessing continuous derivatives.

The components of displacement are obtained by integrating the strain displacement relations. Only those terms in the series expansions for the displacements which dominate



near the crack tip are of immediate interest. The stress-strain relations and the series expansion for Young's modulus are used to derive the strain components in terms of the functions  $F(\theta)$ ,  $G(\theta)$ ,  $H(\theta)$ . After some manipulations, the expressions for the Cartesian components of displacement given in terms of the cylindrical coordinates  $(r, \theta)$  appear as

$$u_x \cong \frac{K_I}{\mu_0} \left( \frac{r}{2\pi} \right)^{1/2} g'_x(\theta) + \frac{K_{II}}{\mu_0} \left( \frac{r}{2\pi} \right)^{1/2} g''_x(\theta) + u_{x0} - \omega_0 r \sin \theta + \frac{r}{2\mu_0} \{C_2(\kappa + 1) \cos \theta + D_2(\kappa + 1) \sin \theta\} + O(r^{3/2}) + \dots \quad (23)$$

$$u_y \cong \frac{K_I}{\mu_0} \left( \frac{r}{2\pi} \right)^{1/2} g'_y(\theta) + \frac{K_{II}}{\mu_0} \left( \frac{r}{2\pi} \right)^{1/2} g''_y(\theta) + u_{y0} + \omega_0 r \cos \theta + \frac{r}{2\mu_0} \{C_2(\kappa - 3) \sin \theta - D_2(\kappa + 1) \cos \theta\} + O(r^{3/2}) + \dots \quad (24)$$

where  $\mu_0 = E_0/2(1 + \nu)$ , and  $\kappa = 3 - 4\nu$  for plane strain or  $3 - \nu/1 + \nu$  for generalized plane stress. The constants  $u_{x0}$ ,  $u_{y0}$ , and  $\omega_0$  are associated with rigid body displacements and rotations. The functions  $g'_x(\theta)$ ,  $g''_x(\theta)$ , etc. are given in [14]. The elastic modulus appearing in the leading term of (23) and (24) is  $\mu_0$ , the value of the shear modulus at the crack tip. Again, the nature of the near tip displacement field is the same as for the homogeneous material. The terms  $O(r^{3/2})$  and above do not exhibit the same spatial dependence as the corresponding terms for a homogeneous material. In summary, the form of the terms proportional to  $r^{1/2}$ ,  $r^0$  and  $r$  are "universal" in that they do not depend on variation of the elastic moduli.

The analysis presented above did not include a possible spatial variation in Poisson's ratio. If such a variation were considered, the following substitutions would be made in (23) and (24);  $\mu_0 = E_0/2(1 + \nu_0)$  and  $\kappa = \kappa_0$ .

### 3. Domain-independent integral for nonhomogeneous materials

By far the most common concern pertaining to linear elastic fracture mechanics analysis is the accurate prediction of stress intensity factors in arbitrarily shaped cracked bodies. Rarely can a complete solution for a finite body containing a crack be found using analytical methods. Numerical techniques (e.g. finite elements, finite differences, boundary integrals, etc.) are generally necessary to compute requisite field quantities. After stress and displacement fields have been obtained, two choices exist for extracting the stress intensity factors from the numerical data. They can be calculated "directly" using the computed displacements and/or stresses near the crack tip. The second choice involves calculating the stress intensity factors "indirectly" via path independent integrals. The motivation for using such integrals is that evaluation of near tip field quantities can be avoided, yet the value of the integrals is related to the crack tip stress intensity factors. A balance law associated with linear elasticity will be used to derive integrals which prove useful for computing stress

intensity factors in nonhomogeneous bodies containing cracks. These integrals are also shown to be related to the energetics of the fracture process in certain circumstances.

There are a number of methods to generate path-independent integrals in elasticity theory. The most elegant method is based on Noether's theorem (Knowles [15]). For the present purpose, a simple and direct approach is taken which extends some of the classical work of Eshelby [16]. A nonhomogeneous elastic body subjected to a two-dimensional deformation field (plane strain, generalized plane stress) possesses a strain energy density function  $W$  defined by

$$W = W(\varepsilon_{ij}, x_i) \quad \text{where} \quad \sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \quad (i, j = 1, 2). \quad (25)$$

Note that when the material is homogeneous, the strain energy function will be  $W = W(\varepsilon_{ij})$ . To derive a balance law, the gradient of  $W$  is formed, i.e.,

$$\frac{\partial W}{\partial x_k} = \frac{\partial W}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial x_k} + \left( \frac{\partial W}{\partial x_k} \right)_{\text{expl}} \quad (26)$$

where the "explicit" derivative of  $W$  is defined by

$$\left( \frac{\partial W}{\partial x_k} \right)_{\text{expl}} = \frac{\partial}{\partial x_k} W(\varepsilon_{ij}, x_i) \Big|_{\varepsilon_{ij} = \text{const}, x_m = \text{const for } m \neq k}. \quad (27)$$

Equation (26) becomes

$$W_{,k} = \sigma_{ij} \varepsilon_{ij,k} + (W_{,k})_{\text{expl}}. \quad (28)$$

Using the symmetry properties of the stress tensor, the linearized strain-displacement relations, and the equilibrium equations, it follows that

$$(W \delta_{jk} - \sigma_{ij} u_{i,k})_{,j} - (W_{,k})_{\text{expl}} = 0 \quad (29)$$

where  $\delta_{jk}$  is the Kronecker delta. This vector equation represents a *balance law*, valid pointwise, for a nonhomogeneous elastic body (not necessarily isotropic).

An integral form of (29) proves necessary for application in computational work. A simple closed curve  $\Gamma$  in the  $x_1, x_2$  plane is introduced along with the domain  $\Omega$  which it encloses. By integrating (29) over the domain  $\Omega$  and applying the divergence theorem, the following formula results

$$\oint_{\Gamma} (W n_k - \sigma_{ij} n_j u_{i,k}) d\Gamma - \int_{\Omega} (W_{,k})_{\text{expl}} d\Omega = 0 \quad (30)$$

where  $n_j$  is a measure number of the outward unit normal vector to  $\Gamma$ .

For a linear elastic nonhomogeneous material the strain energy function is of the form

$$W = \frac{1}{2} c_{prst} (x_1, x_2) u_{p,r} u_{s,t} \quad (31)$$

where  $c_{prst}$  denotes the elasticity tensor. For an isotropic nonhomogeneous material, the "explicit" derivative of  $W$  is then

$$(W_{,k})_{\text{expl}} = \frac{1}{2}[\lambda_{,k} \delta_{pr} \delta_{st} + \mu_{,k} (\delta_{ps} \delta_{rt} + \delta_{pt} \delta_{rs})] u_{p,r} u_{s,t} \quad (32)$$

where  $\lambda$  and  $\mu$  are the Lamé' moduli.

So far no mention of a crack in the region  $\Omega$  has been made. However, in applying the divergence theorem it was tacitly assumed that field quantities were continuous, bounded, and generally differentiable on  $\Omega$ . Since the stress and strain fields are singular at a crack tip, and therefore unbounded, the region  $\Omega$  referred to in (30) must not contain a crack tip. In order to derive an integral expression which is valid in the presence of a crack tip, a rather special region  $\Omega$  must be considered. Figure 1 shows a crack located in a two-dimensional body of arbitrary shape. Traction are permitted to act on the crack surfaces. The horseshoe shaped region  $\Omega$  (free of singularities) is bounded by a closed curve  $\Gamma$  composed of segments  $\Gamma_0, \Gamma_c^+, \Gamma_\varepsilon, \Gamma_c^-$  as shown. The region between  $\Gamma_\varepsilon$  and the crack surfaces is  $\Omega_\varepsilon$ . The region  $\Omega_0$  is defined as  $\Omega + \Omega_\varepsilon$ . The divergence theorem can be applied in the region  $\Omega$ , because the singular fields at the crack tip are excluded. Equation (30) can be used to write

$$\oint_{\Gamma_0} b_k d\Gamma + \int_{\Gamma_c^+} b_k d\Gamma + \oint_{\Gamma_\varepsilon} b_k d\Gamma + \int_{\Gamma_c^-} b_k d\Gamma - \int_{\Omega} (W_{,k})_{\text{expl}} d\Omega = 0 \quad (33)$$

where

$$b_k = W n_k - \sigma_{ij} n_j u_{i,k}. \quad (34)$$

If the direction of integration is reversed on the third term of (33), and the region  $\Omega$  is decomposed into  $\Omega_0 - \Omega_\varepsilon$  it follows that

$$\oint_{\Gamma_0} b_k d\Gamma - \int_{\Omega_0} (W_{,k})_{\text{expl}} d\Omega + \int_{\Gamma_c^+} b_k d\Gamma + \int_{\Gamma_c^-} b_k d\Gamma = \oint_{\Gamma_\varepsilon} b_k d\Gamma - \int_{\Omega_\varepsilon} (W_{,k})_{\text{expl}} d\Omega. \quad (35)$$

A vector  $\mathbf{J}^*$  is introduced whose measure numbers are defined by the right-hand side of (35)

$$\mathbf{J}_k^* \equiv \lim_{\Gamma_\varepsilon \rightarrow 0} \left[ \oint_{\Gamma_\varepsilon} b_k d\Gamma - \int_{\Omega_\varepsilon} (W_{,k})_{\text{expl}} d\Omega \right] \quad (36)$$

As the loop  $\Gamma_\varepsilon$  is shrunk onto the crack tip, the domain integral in (36) vanishes, for the following reason. Field quantities are assumed to be expressed in terms of a cylindrical coordinate system whose origin is fixed at the crack tip. Derivatives of the elastic moduli are assumed to be bounded at the crack tip, i.e.,  $\lambda_{,k}$  and  $\mu_{,k}$  are  $O(r^\alpha)$ , where  $\alpha \geq 0$ . Therefore, near the crack tip,  $(W_{,k})_{\text{expl}}$  is  $O(r^{-1+\alpha})$ , and the domain integral vanishes in the limit. Then (35) and (36) can be combined to produce

$$\begin{aligned} \mathbf{J}_k^* &\equiv \lim_{\Gamma_\varepsilon \rightarrow 0} \oint_{\Gamma_\varepsilon} [W n_k - \sigma_{ij} n_j u_{i,k}] d\Gamma \\ &= \lim_{\Gamma_\varepsilon \rightarrow 0} \left\{ \oint_{\Gamma_0} [W n_k - \sigma_{ij} n_j u_{i,k}] d\Gamma - \int_{\Omega_0} (W_{,k})_{\text{expl}} d\Omega \right. \\ &\quad \left. + \int_{\Gamma_c^+} [W n_k - \sigma_{ij} n_j u_{i,k}] d\Gamma + \int_{\Gamma_c^-} [W n_k - \sigma_{ij} n_j u_{i,k}] d\Gamma \right\}. \end{aligned} \quad (37)$$

