in Fig. 5. As shown, smaller magnitudes of the control force consume less fuel. Furthermore, the window size must be greater than one tenth of a period of oscillation to effectively control the system and greater than one period of oscillation to efficiently control the system. Not shown, as the window size increases, the time to reduce the energy in the system increases. The effect of the maximum velocity coefficient $C_v$ on the fuel is shown in Fig. 6. As shown, the neighborhood of maximum velocity decreases as $C_v$ increases, leading to greater efficiency and a reduction in the fuel. However, as $C_v$ increases the time required to reduce the energy increases as shown in Fig. 7 and for $C_v = 1.3$ the time required to reduce the energy becomes unbounded so $C_v$ is bounded above by 1.3. Also shown in Fig. 7, as $C_v$ decreases the neighborhoods of minimum displacement decrease, which also leads to greater efficiency but with an increase in time to reduce energy.

**Distributed Systems**

As stated earlier, decentralized feedback was shown to provide a minimum fuel solution to the linear optimal control problem, implying that decentralized feedback can provide a minimum fuel solution to the nonlinear (on-off) optimal control problem. While the previous section considered single-degree-of-freedom systems, this section shows how to control distributed systems using an on-off decentralized feedback approach.

The partial differential equation governing the motion of a distributed system can be expressed in the form

$$\frac{\partial^2 u}{\partial t^2} = -Lu + f$$

with

- $\rho = \rho(P)$ — mass density at point $P$ in the domain $D$ of the system
- $u = u(P,t)$ — displacement of point $P$ at time $t$
- $L$ — self-adjoint positive definite (or semi-definite) differential stiffness operator
- $f = f(P,t)$ — control force density of point $P$ at time $t$

The discrete control forces acting on the system can be represented by the control force density as follows:

$$f = \sum_{s=1}^{N} f_s \delta_s$$

with

- $N$ — number of discrete control forces
- $f_s = f_s(t)$ — discrete control force acting at point $P_s$
- $\delta_s = \delta(P-P_s)$ — spatial Dirac-delta function
Running Averages and Standard Deviations of Displacements and Velocities

As stated in the introduction, nonlinear (on-off) optimal feedback control forces are characterized when the structure's motion is in the neighborhood of its maximum velocity and minimum displacement. These neighborhoods in time are determined using running averages and running standard deviations of displacements and velocities. Letting the function \( f(t) \) represent either a displacement or velocity at time \( t \), we write

\[
\mu_f(t) = \int_{t-T}^{t} \alpha(s,t)f(s)ds, \quad \int_{t-T}^{t} \alpha(s,t)ds = 1
\]

\[
\nu_f^2(t) = \int_{t-T}^{t} b(s,t) (f(s) - \mu_f(s))^2 ds, \quad \int_{t-T}^{t} b(s,t)ds = 1
\]

(1)

with

\( \mu_f(t) \) — running average of \( f \) at time \( t \)

\( \nu_f(t) \) — running standard deviation of \( f \) at time \( t \)

\( \alpha(s,t) \) — weighting function for the running average

\( b(s,t) \) — weighting function for the running standard deviation

Different weighting functions are shown in Fig. 1. The shape of the weighting functions shown do not change in time and they follow the time coordinate \( t \). Mathematically the weighting functions can be expressed in the form \( \alpha(s,t) = \alpha(s-t) \) as the corresponding running averages and standard deviations are stationary. Differentiating (1), we obtain the recursive form of the running average and the running standard deviation of \( f(t) \).

\[
\frac{d\mu_f}{dt} = \int_{t-T}^{t} \frac{\partial \alpha(s,t)}{\partial t} f(t)ds + \alpha(t,t)f(t) - \alpha(t-T,t)f(t-T)
\]

\[
\frac{d\nu_f^2}{dt} = \int_{t-T}^{t} \left( \frac{\partial b(s,t)}{\partial t} (f(s) - \mu_f(s))^2 ds + b(t,t) (f(t) - \mu_f(t))^2 ds \right)

- b(t,t) (f(t) - \mu_f(t-T))^2
\]

(2)

In the case of a rectangular window \( \int_{t-T}^{t} \frac{\partial \alpha(s,t)}{\partial t} f(s)ds = 0 \),

in the case of a linear window \( \int_{t-T}^{t} \frac{\partial \alpha(s,t)}{\partial t} f(s)ds = \alpha \int_{t-T}^{t} f(s)ds \)

and in the case of an exponential window \( \int_{t-T}^{t} \frac{\partial \alpha(s,t)}{\partial t} f(s)ds = -\beta \int_{t-T}^{t} \alpha(s,t)f(s)ds = -\beta \mu_f(t). \)

Throughout the remainder of the paper we shall determine the neighborhoods of maximum velocity and minimum displacement using the recursive form of the average and standard deviation given in (2). Although the use of the linear window and the exponential window can adequately determine the indicated neighborhoods, for simplicity we shall restrict our attention throughout the remainder of the paper with the rectangular window chosen as our weighting function. Furthermore, note for structures not admitting zero frequency modes of vibration, that it suffices to compute the standard deviation and we can let \( \mu_f(t) = 0 \) in (1) and (2).

As an illustration, let \( f(t) = \sin \omega t \) and \( \mu_f(t) = 0 \). The running standard deviations of \( f(t) \) are shown in Fig. 2 for different nondimensionalized rectangular window lengths \( \omega T \). As shown, larger window lengths provide more accurate predictions of the steady-state values of the standard deviation (corresponding to \( T = \infty \)) and larger amounts of time to reach the steady-