On the Nature of Natural Control

This paper examines families of structural control systems and reveals inherent properties that provide the essential motivation behind the theory of Natural Control. It is determined that the associated fuel consumed by the controls is near minimal when the natural frequencies are identical to the controlled modal frequencies, and when the natural modes of vibration are identical to the controlled modes of vibration. Also, by casting the objective to suppress vibration in the form of an exponential stability condition, it is found that vibration is most efficiently suppressed when the modal damping rates are identical to a designer chosen decay rate. The use of a limited number of control forces over distributed control is characterized by a change in fuel consumed by the controls and by a deterioration in the dynamic performance reflected by changes in the modal damping rates. The Natural Control of a space truss demonstrates the results.

Introduction

Whereas control theory pertains to a wide range of problems, structural control theory is more restrictive, pertaining to linear systems often characterized by their self-adjointness and positive semi-definiteness. The associated system parameters are of three types. The first type are called physical parameters and they consist of mass, stiffness, and geometry. The second type are called dynamic performance parameters. These parameters characterize the structural motion. For uncontrolled systems, the natural dynamic performance parameters consist of natural frequencies and associated natural modes of vibration. For controlled systems, the controlled dynamic performance parameters consist of controlled modal frequencies, associated modal damping rates and associated controlled modes of vibration. The physical parameters can be computed from the uncontrolled dynamic performance parameters and vice versa. The third type of parameters are called the control parameters. They consist of control gains and locations of the force actuators and the measurement sensors. A structural control theory, in order to have practical value, must pertain to a class of linear systems that encompasses structural problems. Moreover, it must find a strong connection between the three indicated types of parameters for the purposes of rendering an efficient design process.

In the late 1970's, Meirovitch, Oz and Baruh developed a new method of structural control, known as Independent Modal-Space Control (IMSC). In a series of papers, the authors demonstrated a method exhibiting an impressive list of characteristics (Meirovitch and Oz, 1980b; Oz and Meirovitch, 1980; Meirovitch and Baruh, 1982). Using IMSC, modal damping rates and controlled modal frequencies could be determined arbitrarily and control gains could be computed with relative ease. Other characteristics were also documented (Meirovitch, et al., 1979, 1980a, 1981b, 1983a, 1985; Silverberg, et al., 1985, 1986). In a paper by Meirovitch and Silverberg (1983b) IMSC was shown to preserve certain properties natural to a structure and the term Natural Control was coined. Natural Control referred to a control system that preserves the natural properties of a structure. That is to say that certain natural dynamic performance parameters were set equal to controlled dynamic performance parameters. Using IMSC, the natural modes of vibration were set equal to the controlled modes of vibration. In addition, Silverberg (1986) demonstrated that the natural frequencies could be set equal to the controlled modal frequencies. The control system became more natural as more properties were preserved.

A common question asked by control theory developers and practitioners alike is why “natural”? Indeed, what advantages are derived by choosing an objective such as the preservation of the natural characteristics of a structure? Does “natural” imply that other methods are “unnatural” (Oz et al., 1983)? Are there circumstances in which it is more desirable not to preserve the structure’s natural characteristics? Unfortunately, these questions have not been answered directly. Although, Meirovitch and Silverberg (1983b) proved that Natural Control is globally optimal and Oz (1983) demonstrated that Natural Control is a minimum gain technique, the connection is weak between these properties and the requirement to preserve the natural characteristics of a structure.

This paper examines the nature of Natural Control and answers the question, why “natural”? Families of structural control systems are examined and a comparison between control systems is made based on different expressions for control effort. Because the expressions can involve the absolute values of forces, it would be overly cumbersome to carry out the minimizations analytically. Instead, we resort to the numerical computation of different expressions for control effort.

Following the introduction, the family of single degree of freedom systems is considered. For this family of systems, the controlled frequencies are varied as the damping rates are held fixed. Next, the family of two-degrees-of-freedom systems is considered. For this family of systems the controlled modes of vibration are varied as the modal damping rates and the controlled modal frequencies are held fixed. Then, the implica-
The associated eigenvalue problem is defined as
\[ \omega^2 m \phi_r = k \phi_r , \quad (r = 1, 2) \] (11)
with
\[ \omega_r = r \text{th natural frequency} \]
\[ \phi_r = r \text{th natural mode of vibration} \]

The natural modes of vibration satisfy the orthonormality relations
\[ \phi_s^T m \phi_r = \delta_{s r} , \quad \phi_s^T k \phi_r = \omega_r^2 \delta_{s r} , \quad (r, s = 1, 2) \] (12)
with
\[ \delta_{s r} = \text{kronnecker delta} (\delta_{11} = \delta_{22} = 1, \delta_{12} = \delta_{21} = 0) \]
The two natural modes of vibration can be expressed in terms of a single rotation angle
\[ \phi_1 = m^{-1/2} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} , \quad \phi_2 = m^{-1/2} \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \] (13)
with
\[ \theta = \text{natural rotation angle} \]
For the case in which the system is controlled, the control force vector is governed by the feedback law
\[ f(t) = gx(t) + hx(t) \] (14)
with
\[ g, h = \text{control gain matrices} \]

Introducing (14) into (10) yields the differential equation of motion
\[ m \ddot{x}(t) = (-k + g)x(t) + h\dot{x}(t) \] (15)
The control gain matrices \( g \) and \( h \) are assumed to admit real controlled modes of vibration. The controlled modes of vibration are expressed in terms of a single rotation angle in the form
\[ \psi_1 = m^{-1/2} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} , \quad \psi_2 = m^{-1/2} \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \] (16)
with
\[ \psi_r = r \text{th controlled mode of vibration} \]
\[ \theta = \text{controlled rotation angle} \]
The controlled modes of vibration satisfy the orthonormality relations
\[ \psi_s^T m \psi_r = \delta_{s r} , \quad \psi_s^T (k + g) \psi_r = - (\alpha_r^2 + \beta_r^2) \delta_{s r} , \quad \psi_s^T h \psi_r = -2 \alpha_r \delta_{s r} \] (17)
with
\[ \alpha_r = r \text{th controlled modal damping rate} \]
\[ \beta_r = r \text{th controlled modal frequency} \]
Consider the change of variables in which the controlled system response is expressed as a linear combination of the controlled modes of vibration
\[ x(t) = \psi_1 \xi_1(t) + \psi_2 \xi_2(t) \] (18)
with
\[ \xi_r(t) = r \text{th controlled modal displacement} \]
Premultiplying (15) by \( \psi_r^T m \) and considering (17) yields
\[ \ddot{\xi}_r(t) = \psi_r^T m \ddot{x}(t) , \quad (r = 1, 2) \] (19)
Introducing (18) into (10), premultiplying by \( \psi_r^T \) and considering (17) yields the controlled modal equations
\[ \ddot{\xi}_r(t) = - (\alpha_r^2 + \beta_r^2) \xi_r(t) - 2 \alpha_r \xi_r(t) , \quad (r = 1, 2) \] (20)
The controlled system response is given by
\[ x(t) = \begin{bmatrix} \cos \theta / \sqrt{m_1} \\ \sin \theta / \sqrt{m_2} \end{bmatrix} e^{-\alpha_1 t} \]
\[ \times \left( \xi_{10} \cos \beta_1 t + \frac{\xi_{10} + \alpha_1 \xi_{10}}{\beta_1} \sin \beta_1 t \right) \]
\[ + \left( \frac{-\sin \theta / \sqrt{m_1}}{\cos \theta / \sqrt{m_2}} \right) e^{-\alpha_2 t} \]
\[ \times \left( \xi_{20} \cos \beta_2 t + \frac{\xi_{20} + \alpha_2 \xi_{20}}{\beta_2} \sin \beta_2 t \right) \] (21)
with
\[ \xi_{r0} = \psi_r^T m x(0) = r \text{th initial controlled modal displacement} \]
\[ \dot{\xi}_{r0} = \psi_r^T m x'(0) = r \text{th initial controlled modal velocity} \]
The structural control problem here amounts to establishing the relationships between three sets of parameters, namely, the physical parameters \( m \) and \( k \), the dynamic performance parameters \( (\omega_r, \Phi_r, \alpha_r, \beta_r, \text{and } \psi_r) \) and the control parameters \( g \) and \( h \). The control parameters are related to the other sets of parameters by
\[ g = - \sum_{r=1}^{2} m \psi_r \left( \alpha_r^2 + \beta_r^2 \right) \psi_r^T m + \sum_{r=1}^{2} m \Phi_r \omega_r^2 \psi_r^T m , \]
\[ h = - \sum_{r=1}^{2} m \psi_r \beta_r \psi_r^T m \] (22)
The dynamic performance parameters are of two types; either natural dynamic performance parameters \( (\omega_r \text{ and } \Phi_r) \) or controlled dynamic performance parameters \( (\alpha_r, \beta_r, \text{and } \psi_r) \). From (22) it is apparent that there exists control parameters which yield any desirable controlled dynamic performance; that is the question arises how to choose \( \alpha_r, \beta_r, \text{and } \psi_r \).

Because the interest lies in suppressing vibration, the choice of \( \alpha_r \) depends directly on how fast the designer wishes the motion to subside. On the other hand, the choice of \( \beta_r \) and \( \psi_r \) is less obvious. In the previous section it was determined that choosing the controlled frequency nearly identical to the natural frequency minimizes the control effort (see Property 1). In this section, we will investigate different control systems in which \( \alpha_r \) and \( \beta_r \) are held fixed and we consider different controlled modes of vibration \( \psi_r \), or equivalently we consider different controlled rotation angles \( \theta \). For each of these control systems, the following two suitable expressions for control effort are computed:
\[ F_{u1} = \int_0^\infty \left| f_1^1(t) + f_2^1(t) \right| dt , \]
\[ F_{u2} = \int_0^\infty \left| f_1^2(t) + |f_2^2(t)| \right| dt \] (23)
pulse of magnitude 250 N-sec at node no. 10 in the x direction. The free system response is shown in Fig. 13. The truss structure was controlled at the designer decay rate \( \alpha = 0.1 \) rad/sec. Four control systems were considered in which the following expression for control effort was computed:

\[
F_\tau = \int_0^\infty \int_{r-1}^\infty f^2(x,t)/m(x)dx = \sum_{r=1}^{\infty} Q^2(r)dt
\]

Using control system (1) \( n = 18 \) control forces were considered and the fuel consumed \( F_\tau = 0.1147 \). Using control system (2), \( n = 10 \) control forces were considered and the fuel consumed \( F_\tau = 0.1442 \). Using control system (3), \( n = 4 \) control forces were considered and the fuel consumed \( F_\tau = 0.1967 \). Using control system (4), distributed control forces were considered and the fuel consumed \( F_\tau = 0.0646 \). The truss structure possessed 1 percent structural damping. The locations of the control forces for the first 3 control systems are given in Table 2. The associated lower controlled eigenvalues are given in Table 3. The subdomain masses for each control system are given by \( M_r = M/n, (r = 1, 2, \ldots, n) \) (Ref. 11) The associated controlled system responses are shown in Figs. 14-16.

### Summary

This paper examined families of control systems and, in the process, revealed inherent properties which have provided the essential motivation behind the theory of Natural Control. We distinguished between three types of system parameters, namely, physical parameters, dynamic performance parameters, and control parameters. The problem of efficient control system design is to find a connection between the three types of parameters. This paper finds a connection between the natural dynamic performance parameters (associated with the freely vibrating system) and the controlled dynamic performance parameters. It is determined that the control effort is near minimal when the natural frequencies are identical to the controlled modal frequencies and when the natural modes of vibration are identical to the controlled modes of vibration. Also by casting the objective to suppress vibration in the form of an exponential stability condition, it is determined that vibration is best suppressed by uniformly damping the system, i.e., by setting the controlled modal decay rates equal to the designer decay rate. This leads to a decentralized control law that is independent of the system stiffness. Finally, the use of a limited number of control forces leads to two notable effects. First, the controlled modal decay rates change and, secondly, the fuel consumed by the controls is different than for a control system implemented by distributed controls. These two effects are illustrated in the Natural Control of a space truss structure.

### References


### Table 2 Control forces nodal locations for 3 control systems

<table>
<thead>
<tr>
<th>n</th>
<th>(1)</th>
<th>(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>21</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>26</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>30</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>38</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>46</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>50</td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>28</td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>30</td>
</tr>
<tr>
<td>13</td>
<td>13</td>
<td>36</td>
</tr>
<tr>
<td>14</td>
<td>14</td>
<td>38</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
<td>40</td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>46</td>
</tr>
<tr>
<td>17</td>
<td>17</td>
<td>48</td>
</tr>
<tr>
<td>18</td>
<td>18</td>
<td>50</td>
</tr>
</tbody>
</table>

### Table 3 Controlled eigenvalues \( \lambda_r = -\alpha + i\beta_r \) (rad/sec) for 3 control systems

<table>
<thead>
<tr>
<th>Mode #</th>
<th>( \alpha_r )</th>
<th>( \beta_r )</th>
<th>( \alpha_r )</th>
<th>( \beta_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.27</td>
<td>0.0</td>
<td>0.68</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.26</td>
<td>0.0</td>
<td>0.35</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>0.26</td>
<td>0.0</td>
<td>0.33</td>
<td>0.0</td>
</tr>
<tr>
<td>4</td>
<td>0.11</td>
<td>0.0</td>
<td>0.33</td>
<td>0.0</td>
</tr>
<tr>
<td>5</td>
<td>0.10</td>
<td>0.0</td>
<td>0.23</td>
<td>0.0</td>
</tr>
<tr>
<td>6</td>
<td>0.10</td>
<td>0.0</td>
<td>0.11</td>
<td>0.0</td>
</tr>
<tr>
<td>7</td>
<td>0.10</td>
<td>0.0</td>
<td>0.11</td>
<td>0.0</td>
</tr>
<tr>
<td>8</td>
<td>0.10</td>
<td>0.0</td>
<td>0.10</td>
<td>0.0</td>
</tr>
<tr>
<td>9</td>
<td>0.10</td>
<td>0.0</td>
<td>0.10</td>
<td>0.0</td>
</tr>
<tr>
<td>10</td>
<td>0.10</td>
<td>0.0</td>
<td>0.10</td>
<td>0.0</td>
</tr>
<tr>
<td>11</td>
<td>0.10</td>
<td>0.0</td>
<td>0.10</td>
<td>0.0</td>
</tr>
<tr>
<td>12</td>
<td>0.10</td>
<td>0.0</td>
<td>0.10</td>
<td>0.0</td>
</tr>
</tbody>
</table>

---

416 / Vol. 111, OCTOBER 1989

Transactions of the ASME
Fig. 12 Elastic modes of vibration of truss structure

Fig. 13 Elastic displacements at node 41 for the uncontrolled system
Fig. 16. Elastic displacements at node 41 for the controlled system, given actuators at 4 nodes.