The external moment acting about the origin is given by
\[
\mathbf{M}(t) = \int_D \mathbf{u}_C(P) \times \mathbf{f}_T(P,t) \, dD = \int_D \mathbf{u}_C(P) \times \mathbf{f}_T(P,t) \, dD \\
+ \int_D \mathbf{u}_G(P) \times \mathbf{f}_E(P,t) \, dD + \int_D \mathbf{u}_E(P) \times \mathbf{f}_E(P,t) \, dD
\]  
(28)

Considering the identities in Eqs. (23) and (24), we obtain
\[
\int_D \mathbf{u}_C(P) \times \mathbf{f}_T(P,t) \, dD = 0 
\]  
(29a)
\[
\int_D \mathbf{u}_G(P) \times \mathbf{f}_E(P,t) \, dD = 0
\]  
(29b)

Therefore, the translational component and the elastic component of the force produce no external moment about the mass center. Then, from Eqs. (28) and (29), the external moment \(\mathbf{M}(t)\) is given by
\[
\mathbf{M}(t) = \int_D \mathbf{u}_G(P) \times \mathbf{f}_E(P,t) \, dD
\]  
(30)

The external linear momentum of the spacecraft is given by
\[
\mathbf{P}(t) = \int_D \mathbf{u}(P) \mathbf{g}(P,t) \, dD = \int_D \mathbf{u}(P) \mathbf{g}_T(P,t) \, dD \\
+ \int_D \mathbf{u}(P) \mathbf{g}_G(P,t) \, dD + \int_D \mathbf{u}(P) \mathbf{g}_E(P,t) \, dD
\]  
(31)

Considering the identities in Eqs. (21) and (22), we obtain
\[
\int_D \mathbf{u}(P) \mathbf{g}_G(P,t) \, dD = 0 
\]  
(32a)
\[
\int_D \mathbf{u}(P) \mathbf{g}_E(P,t) \, dD = 0
\]  
(32b)

Therefore, the rotational component and the elastic component of the motion produce no external linear momentum. Then, from Eqs. (31) and (32), the external linear momentum is given by
\[
\mathbf{P}(t) = \int_D \mathbf{u}(P) \mathbf{g}_T(P,t) \, dD
\]  
(33)

The external angular momentum about the origin is given by
\[
\mathbf{H}(t) = \int_D \mathbf{u}(P) \mathbf{u}_C(P) \times \mathbf{g}(P,t) \, dD = \int_D \mathbf{u}(P) \mathbf{u}_C(P) \times \mathbf{g}_T(P,t) \, dD \\
+ \int_D \mathbf{u}(P) \mathbf{u}_G(P) \times \mathbf{g}_G(P,t) \, dD \\
+ \int_D \mathbf{u}(P) \mathbf{u}_E(P) \times \mathbf{g}_E(P,t) \, dD
\]  
(34)

Considering the identities in Eqs. (23) and (24), we obtain
\[
\int_D \mathbf{u}(P) \mathbf{u}_G(P) \times \mathbf{g}_G(P,t) \, dD = 0 
\]  
(35a)
\[
\int_D \mathbf{u}(P) \mathbf{u}_E(P) \times \mathbf{g}_E(P,t) \, dD = 0
\]  
(35b)

Therefore, the translational component and the elastic component of the motion produce no angular momentum about the mass center. Then, from Eqs. (34) and (35), the external angular momentum about the origin is given by
\[
\mathbf{H}(t) = \int_D \mathbf{u}(P) \mathbf{u}_C(P) \times \mathbf{g}_T(P,t) \, dD
\]  
(36)

III. Dynamics of Freely Maneuvering Flexible Spacecraft

In the previous section, the equations describing the dynamics of freely nonmaneuvering flexible spacecraft were derived. Using a standard "small-motion" assumption, a set of three linear partial differential equations of motion was derived and, to simplify the problem, natural decompositions of the motion and of the force were introduced. One question that arises is whether these natural decompositions of the motion and force can be considered in conjunction with the maneuvering flexible spacecraft.

It turns out that a decomposition of the motion and of the force can be considered in conjunction with maneuvering flex-

ible spacecraft when a particular coordinate system is introduced. This coordinate system can track the translational and rotational motions of the spacecraft. Indeed, a tracking coordinate system is introduced that coincides with the rigid-body motion of the spacecraft. Note that when the spacecraft is rigid, the tracking coordinate system degenerates to a body-fixed coordinate system.

Let the origin \( C \) of the tracking coordinate system be located at the spacecraft mass center (not attached to the spacecraft). Then, the position vector of the origin \( C \) relative to the origin \( 0 \) of the inertial coordinate system is denoted by \( \mathbf{u}_0(t) = \mathbf{u}_0(t)\hat{\mathbf{t}}_1 + \mathbf{u}_0(t)\hat{\mathbf{t}}_2 + \mathbf{u}_0(t)\hat{\mathbf{t}}_3 \), where \( \hat{\mathbf{t}}_1, \hat{\mathbf{t}}_2, \) and \( \hat{\mathbf{t}}_3 \) are unit vectors of the inertial coordinate system. The position vector of any point \( P \) on the undeformed spacecraft relative to \( C \) is denoted by \( \mathbf{u}_C(P) = x_1\hat{\mathbf{t}}_1 + x_2\hat{\mathbf{t}}_2 + x_3\hat{\mathbf{t}}_3 \), where \( \hat{\mathbf{t}}_1, \hat{\mathbf{t}}_2, \) and \( \hat{\mathbf{t}}_3 \) are unit vectors of the tracking coordinate system. Because the tracking coordinate system coincides with the rigid-body motion, the displacement of point \( P \) on the spacecraft relative to the undeformed position of point \( P \) observed in the tracking coordinate system represents an elastic displacement denoted by \( \mathbf{u}_E(P,t) = \mathbf{u}_E(P,t)\hat{\mathbf{t}}_1 + \mathbf{u}_E(P,t)\hat{\mathbf{t}}_2 + \mathbf{u}_E(P,t)\hat{\mathbf{t}}_3 \).

Whereas the elastic motion is small, the rigid-body motion is arbitrarily large. The inertial position vector of point \( P \) relative to the origin \( 0 \) of the inertial coordinate system is given by (see Fig. 2)
\[
\mathbf{u}(P,t) = \mathbf{u}_0(t) + \mathbf{u}_C(P) + \mathbf{u}_E(P,t)
\]  
(37)

The inertial displacement vector of point \( P \) is differentiated in time to obtain the inertial velocity vector of point \( P \)
\[
\frac{d}{dt} \mathbf{u}(P,t) = \frac{d}{dt} \mathbf{u}_0(t) + \frac{d}{dt} \mathbf{u}_C(P) + \frac{d}{dt} \mathbf{u}_E(P,t)
\]  
(38)
in which
\[
\frac{d}{dt} \mathbf{u}_C(P) = \mathbf{\Omega}(t) \times \mathbf{u}_C(P)
\]  
(39a)
\[
\frac{d}{dt} \mathbf{u}_E(P,t) = \mathbf{\dot{u}}_E(P,t) + \mathbf{\Omega}(t) \times \mathbf{u}_E(P,t)
\]  
(39b)

where \( d/dt \) represents differentiation in time with respect to inertial coordinates, an overdot represents differentiation in time with respect to tracking coordinates, and \( \mathbf{\Omega}(t) = \Omega_{11}(t)\hat{\mathbf{t}}_1 + \Omega_{12}(t)\hat{\mathbf{t}}_2 + \Omega_{13}(t)\hat{\mathbf{t}}_3 \) denotes the angular velocity vector of the tracking coordinates. The inertial velocity vector of point \( P \) is differentiated in time to obtain the inertial acceleration vector of point \( P \)

![Fig. 2 Maneuvering spacecraft.](image-url)
The elastic restoring forces, in the previous sections, were related to the elastic motion by a differential stiffness operator that was linearized about the static equilibrium of the spacecraft. However, as the angular velocity of the spacecraft increases, the vibration oscillates about a dynamic equilibrium rather than about a static equilibrium. To illustrate the significance of this effect, free-free beams undergoing bending motion, while rotating about a principle axis at a constant angular velocity, will expand longitudinally to a new dynamic equilibrium and then vibrate in bending about an expanded dynamic equilibrium. Then, the moments of inertia associated with the rigid-body motion will increase. Moreover, Sec. VI compares the fundamental frequencies associated with the elastic motion of free-free beams rotating about principle axes in which the stiffness operator is linearized about the static equilibrium, with the dynamic response of these beams linearized about the dynamic equilibrium instead.6-8

II. Dynamics of Freely Nonmaneuvering Flexible Spacecraft

Before proceeding with the development of the equations of motion for the freely maneuvering flexible spacecraft, it is desirable first to consider the equations of motion for the freely nonmaneuvering flexible spacecraft. For convenience, an inertial coordinate system is chosen with an origin that coincides with the undeformed position of the spacecraft mass center. The displacement vector of point P at time t is measured in inertial coordinates and denoted by \( u(P,t) \). The wavy underscore denotes three-dimensional vectors. The undeformed position of point P measured relative to the origin is denoted by \( u_c(P) = x_1 \hat{i}_1 + x_2 \hat{i}_2 + x_3 \hat{i}_3 \) where \( \hat{i}_1, \hat{i}_2, \) and \( \hat{i}_3 \) are unit vectors of the inertial coordinate system (see Fig. 1).

Because the spacecraft is flexible, each point P on the spacecraft is exerted upon by elastic restoring forces. The elastic restoring forces depend on the spacecraft displacement. When the displacement relative to its equilibrium is "small," only the linear part of the relationship between the elastic restoring force and the displacement is retained, and the remaining nonlinear parts are discarded. Denoting the elastic restoring force by \( f(P,t) \), we obtain

\[
\dot{u}(P,t) = -L u(P,t)
\]

(1)

where \( L \) is a self-adjoint, positive semidefinite linear differential matrix operator expressing the spacecraft stiffness. The spacecraft is also exerted upon by external forces at each point \( P \) denoted by \( f(P,t) \). Considering Newton's laws of motion at each point \( P \), we obtain

\[
\rho(P) \ddot{u}(P,t) = - \mathbf{L} u(P,t) + f(P,t)
\]

(2)

where \( \rho(P) \) denotes the mass density of point \( P \), and overdots represent differentiations in time with respect to the inertial coordinate system. Because the spacecraft is freely suspended in space, the displacement is not constrained by external geometric boundary conditions. The external boundary conditions are all natural boundary conditions.

To simplify the problem, we consider the natural decompositions of the spacecraft displacement and of the external force. The displacement is expressed as a series of natural displacements, and the external force is expressed as a series of natural forces. The natural decomposition simplifies the problem because it produces a special correspondence between the natural displacements and the natural forces. To show this, we consider the eigenvalue problem associated with the stiffness operator, written

\[
\lambda \phi(P) = L \phi(P)
\]

(3)

The solution to Eq. (3), called the eigensolution, is composed of the nonnegative real eigenvalue \( \lambda \) and the associated real eigenfunction \( \phi(P) \). There exists a countable number of eigenvalues, i.e., eigenvalues \( \lambda_r (r = 1, 2, \ldots) \) and associated eigenfunctions \( \phi_r(P) (r = 1, 2, \ldots) \). The eigenfunctions are mutually orthogonal and can be normalized so as to satisfy the two orthonormality relations

\[
\int_0^L \rho(P) \phi_r(P) \cdot \phi_s(P) \, dD = \delta_{rs}, \\
\int_0^L \phi_r(P) \cdot L \phi_s(P) \, dD = \lambda_r \delta_{rs}, \quad (r, s = 1, 2, \ldots)
\]

(4a, 4b)

where \( \delta_{rs} \) is the Kronecker-Delta function, and the integration is carried out over the domain \( D \) of the spacecraft.9

The displacement and the external force are contained in the vector space generated by the eigenfunctions. Therefore, we can express the motion and the external force as linear combinations of the eigenfunctions as

\[
u(P,t) = \sum_{s=1}^{\infty} u_s(P,t), \quad u_s(P,t) = \phi_s(P) u_s(t)
\]

(5)

\[
\mathbf{f}(P,t) = \sum_{s=1}^{\infty} f_s(P,t), \quad f_s(P,t) = \rho(P) \phi_s(P) f_s(t)
\]

(6)

where \( u_s(P,t) \) and \( f_s(P,t) \) denote natural displacements and natural forces, respectively. The natural displacement \( u_s(P,t) \) has spatial-dependence (shape) \( \phi_s(P) \) identical to the eigenfunction called the natural mode of vibration, and it has time-dependence \( u_s(t) \) called the modal displacement. The natural force \( f_s(P,t) \) has spatial dependence \( \rho(P) \phi_s(P) \) called the natural mode of force, and it has time-dependence \( f_s(t) \) called the modal force. The modal displacements and the modal forces are related to the displacement and to the external force, respectively, by

\[
u_s(t) = \int_0^D \rho(P) \phi_s(P) \cdot u(P,t) \, dD
\]

\[
f_s(t) = \int_0^L \phi_s(P) \cdot f(P,t) \, dD, \quad (r = 1, 2, \ldots)
\]

(7)

Substituting the decompositions of the motion [Eq. (5)] and of the force [Eq. (6)] into the equations of motion (2) while considering the eigenvalue problem [Eq. (3)], the equations for each natural motion are expressed as

\[
\rho(P) \ddot{u}_s(P,t) = -\lambda_s \rho(P) u_s(P,t) + \int_0^L f_s(P,t) \, dD, \quad (s = 1, 2, \ldots)
\]

(8)

Equations (8) are independent. Therefore, the correspondence produced by the natural decomposition is that the \( r \)th natural force can excite the \( r \)th natural displacement and no other natural displacements. Substituting the decompositions [Eqs.