\[ C_r = \lambda_r + g_{rr}/2, \quad R_r = \sum_{s=1}^{m} |g_{rs}| \]

(26)

Note that the centers \( C_r \) are also first-order approximations of the eigenvalues associated with the ideal control system. Equation (26) can be used in order to compare the performance of the control system design with the performance of the ideal control system.

IV. Digitization of the Controls

In the previous section, distributed controls were discretized in space leading to the implication of the controls using a limited number of control forces. The controls acted continuously in time. The controls can also be discretized in time leading to digital controls. In the process, the dynamic parameters of the controls are expected to change depending on the level of digitization. The question arises, at what level of digitization will the dynamic performance of the spacecraft vary significantly from the dynamic performance of the spacecraft with an ideal control system? Consider the continuous controls acting at the \( r \)th node with the associated control law

\[ F_r(t) = -2a\dot{m}_r x_r(t) - a^2\ddot{m}_r x_r(t) \]  

(27)

Here, \( m_r \) refers to the mass of the region within which the control force \( F_r(t) \) acts.11 Over a small time increment \( T \), we apply an impulse

\[ I_r(t) = \int_{t}^{t+T} F_r(t) \, dt = F_r(t)T \]

(28)

so that

\[ I_r(t) = -2a\dot{m}_r x_r(t) - a^2\ddot{m}_r x_r(t) \]  

(29)

Instead of applying continuously acting controls as suggested by Eq. (27), let us apply an impulse every \( kT \) seconds. Then, we replace the continuous control law (27) with the digital control law

\[ I_r(t) = -2ak\dot{m}_r x_r(t) - a^2k\ddot{m}_r x_r(t) \]  

(30)

where the impulse \( I_r(t) \) is applied every \( kT \) seconds. The particular effects of implementing Eq. (30) rather than (27) are described in the numerical example.

V. Uniform Damping of a Simply Supported Beam

As an illustrative example, we control a simply supported beam of length \( a = 10 \) units with unit mass per unit length and unit stiffness density. For this simple example, the equations of motion admit closed-form expressions. The normalized eigenfunctions and natural frequencies are

\[ \psi_r(x) = (2/a)^{1/2} \sin \left( \frac{r\pi x}{a} \right) \omega_r = \left( \frac{r\pi}{a} \right)^2, \]

\[ r = 1, 2, \ldots, m \]  

(31)

For the sake of this example, we assume that the lowest \( m = 10 \) modes of vibration contribute significantly to the overall system response and that the contribution of the remaining modes to the motion is negligible. The beam is given an initial unit step input at \( x = 0 \) for 2 seconds. We design for a uniform exponential decay rate of \( \alpha = 1.0 \) and we assume that 12 structural damping is present in the beam.

As a first step, the ideal control system is designed. The free response is shown in Figure 1 and the ideal control system response is shown in Figure 2. The ideal controller eigenvalues are given in Table 1. Next we consider implementing the control system using a discrete number of control forces. In order to approximate the ideal control system, we locate control forces along the beam at the points \( F_r, (r = 1, 2, \ldots, s; \ s = 4, 5) \) (see Table 2). The associated control laws are given by

\[ F_r(t) = -2a\dot{m}_r \dot{x}_r(t) - a^2m_r \ddot{x}_r(t), \quad m_r = a/s, \]

(32)

where \( x_r(t) \) is the displacement at \( F_r \). Here, again, \( m_r \) represents the mass in the region of the \( r \)th control force. The responses of the beam with the discrete controls are shown in Figures 3 and 4. The corresponding fuels consumed by the controls are shown in Figures 5 and 6. Also, the corresponding first-order approximations of the closed-loop eigenvalues are given in Tables 3 and 4.

Next, we digitize the control law (33). The responses of the beam using digitized discrete controls are shown in Figures 7 and 8. The corresponding fuels consumed by the controls are shown in Figures 9 and 10.

VI. Conclusions

A control system design approach for flexible spacecraft has been presented. The control system design is carried out in two steps. The first step consists of determining an "ideal" uniform exponential rate at which we desire the spacecraft motion to dampen. Next, we construct a control with dynamic performance that is close to the "ideal" using a limited number of actuators. It is also shown that the controls can be digitized when it is desirable to create forces using impulses.

The control system design approach is demonstrated with a simple numerical example in which it is shown that close to ideal dynamic performances can be obtained with a relatively small number of actuators. Also, the effect of digitizing the controls on the dynamic performance is illustrated.

References


where $\delta_{rs} = 0$ for $r \neq s$ and $\delta_{rr} = 1$. We express the displacement vector $\chi(t)$ as a linear combination of the lower $m$ modes, written

$$\chi(t) = \phi_1 u_1(t) + \phi_2 u_2(t) + \ldots + \phi_m u_m(t)$$

(4)

where $m < n$, and $u_r(t) \ (r = 1, 2, \ldots, m)$ are modal displacements which express the degree to which the modes participate in the system response. Generally, the higher modes do not contribute significantly in the response so they are not included in Eq. (4). The modal displacements are governed by the scalar equations,

$$\ddot{u}_r(t) + \omega_r^2 u_r(t) = f_r(t), \quad (r = 1, 2, \ldots, m)$$

(5)

where the modal forces $f_r(t)$ are related to the modal force $F(t)$ by

$$f_r(t) = \Phi_r^T F(t), \quad (r = 1, 2, \ldots, m)$$

(6)

We have assumed here that the modes are normalized, i.e., that Eq. (3) is satisfied. It remains to compute the modal displacements in Eq. (6). Let us first distinguish between rigid-body modes for which $\omega_r = 0$ and flexible-body modes for which $\omega_r \neq 0$.

(A) Rigid-body Modal Response ($\omega_r = 0$)

We rewrite Eq. (5) in the state space by introducing the change of variables

$$\mathbf{u}_r(t) = [u_r(t) \ u_r(t)]^T$$

and obtain the modal equations

$$\dot{\mathbf{u}}_r(t) = A \mathbf{u}_r(t) + B f_r(t)$$

(7)

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(8)

The solution to Eq. (8) can be converted into a difference equation. Letting $T$ denote the time step, and letting $u_r(k)$ and $v_r(k)$ denote the $r$th modal velocity and modal displacement at time $kT$, $(k = 0, 1, 2, \ldots)$ we obtain the difference equations

$$u_r(k+1) = u_r(k) + T v_r(k)$$

(9a)

$$v_r(k+1) = \frac{T^2}{2} v_r(k) + u_r(k) + \frac{T^2}{2} f_r(k)$$

(9b)

Equation (9) is used to compute the response of a rigid-body mode.

(B) Flexible-body Modal Responses ($\omega_r \neq 0$)

Equation (5) describes the motion of an undamped oscillator. However, structures experience small degrees of structural damping. We can introduce some damping into the mathematical model at the exponential rate $\alpha_r = 2 \Xi \omega_r$ by replacing Eq. (5) with

$$\ddot{u}_r(t) + 2 \alpha_r \dot{u}_r(t) + (\alpha_r^2 + \omega_r^2) u_r(t) = f_r(t)$$

(10)

The natural frequency in Eq. (10) is identical to that in Eq. (5). We rewrite Eq. (10) by introducing the change of complex variables

$$u_r(t) = \text{Re}(\omega_r(t)), \quad \dot{u}_r(t) = \text{Re}(i \omega_r(t))$$

where $\lambda_r = -\alpha_r + i \omega_r$, and we obtain the complex modal state equations

$$\dot{u}_r(t) = \lambda_r u_r(t) + f_r(t)/(i \omega_r)$$

(11)

Letting $T$ denote the time step, and letting $\omega_r(k)$ denote the complex modal state at time $kT$, the response to Eq. (11) is given by the difference equation

$$u_r(k+1) = \Phi_r \omega_r(k) + f_r(k)$$

(12)

where

$$\Phi_r = e^{\lambda_r T}, \quad \Gamma_r = (\Phi_r - 1)/(i \alpha_r \omega_r)$$

(13)

Equation (12) is used in order to compute the response of a flexible-body mode. For these purposes, it is desirable to take a time step smaller than one tenth of the smallest flexible body period of oscillation.

III. Control System Design

The control system design is carried out in two steps. In the first step, one constructs the "ideal" control system with the best dynamic performance that nature will allow. Such a system requires distributed forces which are certainly impractical for most applications. The second step consists of designing a control system of minimal cost and greatest simplicity and one which imitates the ideal control system. Perhaps the simplest way to carry out the second step is to consider various designs and to compare the dynamic performances of these designs with the dynamic performance of the ideal control system.

Step 1: The ideal control system

For vibration suppression, pointing, and shape control, the ideal control system is one which damps all the modes of vibration at a single exponential rate $\alpha R^1 R^1$. The linear feedback control law is

$$F(t) = 2 \alpha R g(t) - \alpha^2 R g(t)$$

(14)

Substituting Eq. (4) into (14) while considering the orthonormality conditions (3), we obtain the expressions for the modal control forces