Comment on "Conjecture About Orthogonal Functions"

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Introduction

The conjecture of Ref. 1 may be paraphrased as follows. Assume a set of real functions \( \{\phi_0, \phi_1, \ldots\} \), orthonormal with respect to a weighting function \( w(x) \) on a finite interval \([a, b]\). Assume also that, for each \( n > 0 \), the \( n \)th orthogonal function has \( n \) simple zeros \( \{x_1, x_2, \ldots, x_n\} \) in the interior of \([a, b]\). Then there exists a set of weights \( \{\omega_0, \omega_1, \omega_2, \ldots, \omega_n\} \) so that the following is an orthonormal basis for \( \Psi^r \):

\[
\psi_r = [\omega_0 \phi_r - 1(x_1), \omega_2 \phi_r - 1(x_2), \ldots, \omega_n \phi_r - 1(x_n)]^T, \quad 1 \leq r \leq n
\]

The conjecture is equivalent to the following matrix relations:

\[
\Phi = \begin{bmatrix}
\phi(x_1) & \phi_1(x_1) & \ldots & \phi_0 - 1(x_1) \\
\phi_0(x_1) & \phi_1(x_1) & \ldots & \phi_0 - 1(x_1) \\
\phi_0(x_n) & \phi_1(x_n) & \ldots & \phi_0 - 1(x_n)
\end{bmatrix},
\Omega = \begin{bmatrix}
\omega_0^2 & 0 & \ldots & 0 \\
0 & \omega_2^2 & \ldots & 0 \\
& & \ldots & \omega_n^2
\end{bmatrix}
\]

\[
I = \Phi^T \Omega \Phi, \quad \Omega = (\Phi^T)^{-1} (\Phi)^{-1} = (\Phi \Phi^T)^{-1}, \quad \Omega^{-1} = \Phi \Phi^T
\]

Because the inverse of a diagonal matrix must be diagonal, the conjecture says that the \( n \) vectors must be orthogonal, and the weights simply normalize them.

The conjecture is not true in general but is true for orthonormal polynomials. In fact, for orthonormal polynomials on the finite interval, the conjecture is implied by the Gauss quadrature formula (Ref. 2, pp. 18, 19) and the weights are the square roots of the Christoffel numbers.

Gauss Quadrature Formula

Given \( \{p_0, p_1, \ldots\} \), orthonormal polynomials on \([a, b]\) with weighting function \( w(x) \), where the polynomials are ordered by their degree, then the zeros of each polynomial are distinct and located in the interior of \([a, b]\) (Ref. 2, p. 14). For any polynomial \( f(x) \) of degree \( 2n - 1 \) or less and the roots \( \{x_1, x_2, \ldots, x_n\} \) of the \( n \)th orthogonal polynomial, the Gauss quadrature formula, where the constants \( \lambda_k \) are known as the Christoffel numbers, is

\[
\int_a^b w(x)f(x)dx = \sum_{k=1}^n f(x_k)\lambda_k, \quad \frac{1}{\lambda_k} = \frac{1}{\sum_{k=0}^{n-1} \int_a^b p_k^2(x)w(x)dx}
\]

In particular, the product of two orthonormal polynomials of degree less than \( n \) is certainly a polynomial of degree less than \( 2n - 1 \), so

\[
\int_a^b w(x)p_r(x)p_s(x)dx = \sum_{k=1}^n p_r(x_k)p_s(x_k)\omega_k^2 = \delta_{rs}
\]

\[
\omega_k = \sqrt{\lambda_k}
\]

Orthogonal Polynomials

In Table 1 of Ref. 1, cases 1, 2, 4, and 7 are all sets of orthogonal polynomials, and so the Gauss quadrature formula applies. For the three cases that are not explicitly listed as polynomials, the substitution \( y = \cos x \) gives sets of orthonormal polynomials.

For example, substituting into the set \( 1, \sqrt{2} \sin k\pi x \), we obtain the weighting function \( w(y) = (2(1 - y^2))^{1/2} \) and the polynomials \( 1, 2y, 3y^2 - 1, \ldots \) for \( y \) in the interval \([-1, 1]\).

Nonharmonic Sinusoids

The putative verification of the case of nonharmonic sines \( \{\phi_k = \sin \beta_k x\} \) given in Eqs. (11a–14) and the subsequent paragraph of Ref. 1 (p. 199) is in error. Equation (12) gives the zeros of \( \cos \beta_{k+1} x \) rather than of \( \phi_{k+1} \). These numbers are used in Eqs. (13) and (14) to generate weights that are irrelevant to the stated problem. Note that the zeros of \( \sin \beta_k x \) are listed correctly in the paragraph following Eq. (14).

The conjecture was tested for this case by forming the \( 5 \times 5 \) matrix \( \Phi \) whose columns are the functions \( \phi_k = \sin \beta_k x, \) \( k = 1, 2, \ldots, 5 \), evaluated at the zeros of \( \sin \beta_k x \):

\[
a_k = \sqrt{\frac{2(\beta_k^2 + 4)}{\beta_k^2 + 6}}
\]

and \( \beta_k \) is the solution of \( \beta_k \cot \beta_k + 2 = 0 \).

The values of \( \beta_k \) were found using the MATLAB® fzero function.

<table>
<thead>
<tr>
<th>( \beta_k )</th>
<th>( k = 1, \ldots, 6 )</th>
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<tbody>
<tr>
<td>1.28222570593341</td>
<td>1.36913086305514</td>
</tr>
<tr>
<td>1.40336232749620</td>
<td>1.40745711060934</td>
</tr>
<tr>
<td>2.288927972810340</td>
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<tr>
<td>11.17270586832998</td>
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</table>

zeros:

<table>
<thead>
<tr>
<th>( \beta_k )</th>
<th>( k = 1, \ldots, 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1806214324067</td>
<td>0.36124286648135</td>
</tr>
<tr>
<td>0.72248573296269</td>
<td>0.90310716620337</td>
</tr>
</tbody>
</table>
As Hendry stated, letting $\rho(x) = p_r(x)p_s(x)$ ($r = s = 1, 2, \ldots , n$) in Eq. (1) reduces Eq. (1) to a transformation between the orthogonality condition for functions and the orthogonality condition for $n$-dimensional vectors, which is precisely the conjecture in Ref. 1. It was this conjecture that was used to develop a method of controlling transient vibration in distributed systems.1

Hendry also pointed out for a particular orthogonal set of sinusoidal functions treated in Ref. 1 that the conjecture was violated. It should also be pointed out that, whereas the conjecture was violated for that set of orthogonal functions, the conjecture was still accurate to three decimal places, indicating an additional strength of the Gauss–Jacobi quadrature theorem—that as a practical matter it can be applied to more than orthogonal polynomials.

While on the topic of the Gauss–Jacobi quadrature theorem, it is interesting to observe that letting $\rho(x) = p_r(x)w(x, t)$ in Eq. (1) reduces it to a quadrature of the $r$th modal coordinate, i.e., a modal filter.2 The quadrature is exact if $w(x, t)$ is contained in the $(2n - r)$-dimensional space generated by the orthogonal polynomials.

Finally, the proof of the Gauss–Jacobi quadrature theorem, Eq. (1), is given in the Appendix for the reader’s benefit.5

Appendix: Proof of Gauss–Jacobi Quadrature Theorem

Equation (1) is now derived. As preliminaries, we obtain the orthogonal polynomials $p_0(x), p_1(x), \ldots , p_n(x)$ by orthogonalizing $1, x, x^2, \ldots , x^n$. They are unique provided $p_r(x)$ is a polynomial of precise degree $r$ in which the coefficient of $x^r$ is positive and provided that

$$\int_a^b p_r(x)p_s(x)w(x)\, dx = \delta_{rs} \quad (r, s = 1, 2, \ldots , n)$$

We let $w(x)$ represent a weighting function, assumed to be nonnegative and measurable in the Lebesgue’s sense and such that

$$\int_a^b w(x)\, dx > 0$$

Depending on the limits of integration and on the weighting function, we obtain such classical orthogonal polynomials as Jacobi, Laguerre, Hermite, ultraspherical, Tchebichef of the first and second kinds, and Legendre.

A function in the linear space generated by the orthogonal polynomials up to order $n$ is said to be contained in $\pi_n$. Notice that

$$\int_a^b p_r(x)p_s(x)w(x)\, dx = 0$$

if $p_r(x)$ is in $\pi_{r-1}$, from which it follows that

$$\int_a^b p_r(x)x^s\, dx = 0 \quad (s = 1, 2, \ldots , r - 1)$$

Next, notice that the orthogonal polynomials satisfy the Christoffel–Darboux recursive formula

$$p_r(x) = A_r x + B_r \sum_{s=1}^{r-1} C_r p_s(x) - C_{r-1} p_r(x) \quad (r = 2, 3, \ldots , n)$$

in which $A_r = k_r/k_{r-1} > 0$ and $C_r = \alpha_r/\alpha_{r-1} = k_r/k_{r-1}/k_{r-1}^2 > 0$ and where $k_r$ is the highest coefficient of $p_r(x)$. To prove this, first determine $A_r$ so that $p_r(x) - A_r x p_{r-1}(x)$ lies in $\pi_{r-1}$. Then look at

$$0 = \int_a^b p_r(x)p_{r-1}(x)w(x)\, dx$$

to obtain $C_r$. Next, consider the important identity of the form

$$p_n(x)p_0(y) + p_1(x)p_1(y) + \cdots + p_n(x)p_n(y) = (x/y)\left[ p_n+1(x)p_n(y) - p_n(x)p_{n+1}(y)\right]$$

which follows from the Christoffel–Darboux recursive formula. Also, when $x = y$, notice that this important identity reduces to $p_n(x)p_n(y) + \cdots + p_n(x) = (k_n/k_{n+1})[p_n+1(x)p_n(x) - p_n(x)p_{n+1}(x)]$.

Now we approximate any $p(x)$ in $\pi_{n-1}$ using the Lagrange interpolation polynomial of degree $n - 1$, written...
\[ L(x) = \sum_{u=1}^{n} \frac{\rho(x_u) p_u(x)}{p_u'(x_u)(x - x_u)} \]

Notice that \( p_u(x)/[p_u'(x_u)(x - x_u)] = 1 \) as \( x \) approaches \( x_u \) and \( p_u(x_u)/[p_u(x_u)(x_u - x_u)] = 0 \). It follows that \( L(x_u) = \rho(x_u) \) (\( u = 1, 2, \ldots, n \)).

The Gauss-Jacobi quadrature theorem is now proven by observing that the zeros of \( \rho(x) - L(x) \) are \( x_1, x_2, \ldots, x_n \), implying that it is a product of \( p_u(x) \), that is, \( \rho(x) - L(x) = p_u(x) r(x) \) for some \( r(x) \) in \( \pi_{n-1} \). Then

\[ \int_a^b \rho(x)w(x) \, dx = \int_a^b L(x)w(x) \, dx = \sum_{u=1}^n \Lambda_u \rho(x_u) \]

in which

\[ \Lambda_u = \int_a^b \frac{p_u(x)}{p_u'(x_u)(x - x_u)} w(x) \, dx \quad (u = 1, 2, \ldots, n) \]

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**References**