On Coloation of Actuator-Sensor Pairs
in Decentralized Control

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Introduction

Methods of controlling the transient vibration of structural systems have been widely investigated. Undamped and lightly damped structures, which fall within the classification of self-adjoint systems, are characterized by normal mode vibration. The principle method of controlling such systems adopts the modal control philosophy — that controlling vibration is tantamount to controlling the structure’s normal modes of vibration. When the modes are damped uniformly, the modal control reduces to a decentralized control, in which the need for modal information disappears. Furthermore, under the proper conditions, when the associated actuator-sensor pairs are placed at the nodes of the lowest uncontrolled mode, the resulting decentralized control becomes mode-invariant and frequency-invariant.

The central question examined in this note concerns those situations when it is impractical to precisely colocate the actuators and sensors. When the actuators and sensors are not precisely colocated, can stability be guaranteed? In a sense, how precise do we need to be to refer to an actuator-sensor pair as colocated? Some observations on stability characteristics in the presence of non-colocated actuator-sensor pairs were made elsewhere. This note examines two types of non-colocation: uniform spacing, in which the actuators are uniformly spaced away from the sensors; and nonuniform spacing, in which the sensors are spaced away from the actuators uniformly, except for one actuator-sensor pair. These two types of non-colocation are considered in two examples — a uniform pinned-pinned beam and a uniform cantilevered beam.
Node Control of a Uniform Beam

Consider the motion of a uniform beam described by the partial differential equation

$$\frac{EI}{\partial x^4} \frac{\partial^4 u(x, t)}{\partial x^4} + m \frac{\partial^2 u(x, t)}{\partial t^2} = f(x, t)$$

(1)

where \(u(x, t)\) denotes displacement at point \(x\) at time \(t\), \(m\) denotes mass per unit length, \(EI\) denotes flexural rigidity, and \(f(x, t)\) denotes force per unit length. Let discrete forces \(f_s(t)\) \((e = 1, 2, \ldots, n)\) control the motion of the beam. The associated force per unit length is

$$f(x, t) = \sum_{e=1}^{n} f_s(t) \delta(x - x_{oe})$$

(2)

where \(\delta(x - x_{oe})\) \((e = 1, 2, \ldots, n)\) are unit impulse functions. The motion of the beam is controlled by Node Control (Refs. 3-6). The discrete control forces are governed by the linear state feedback control laws

$$f_s(t) = -g_s u(x_e, t) - h_s \frac{\partial u(x_e, t)}{\partial t} \quad (e = 1, 2, \ldots, n)$$

(3)

where \(g_s\) \((e = 1, 2, \ldots, n)\) and \(h_s\) \((e = 1, 2, \ldots, n)\) denote displacement gains and velocity gains, respectively. Using Node Control, the \(n\) control forces are designed to control the beam’s lowest \(n\) modes of vibration. Toward that end, the actuators and sensors are colocated at the nodes of the \(n+1^{\text{th}}\) mode of vibration, that is \(x_{oe} = x_e \ (e = 1, 2, \ldots, n)\) and \(\phi_{oe}(x_e) = 0 \ (e = 1, 2, \ldots, n)\) where \(\phi_{oe}(x)\) denotes the beam’s \(n+1^{\text{th}}\) mode of vibration.

Although Node Control requires the colocation of actuators and sensors, this can be unfeasible in practice. The question arises as to the precision requirements associated with colocation. Before responding to this question, let us see how the Node Control law (3) controls the lowest \(m\) modes of vibration. First, express the displacement in terms of the lowest \(m\) natural modes of vibration,

$$u(x, t) = \sum_{s=1}^{m} \phi_s(x) \eta_s(t)$$

(4)
where $q_s(t)$ denotes the $s^{th}$ modal displacement. Substitute Eq. (4) into Eq. (1), premultiply the result by $\int_0^L \phi_s(x) \, dx$, and invoke the orthonormality conditions $\int_0^L m \phi_s(x) \phi_s(x) \, dx = \delta_{ss}$ and $\int_0^L EI \phi_s(x) \frac{d^4 \phi_s(x)}{dx^4} \, dx = \omega_r^2 \delta_{ss}$ $(r, s = 1, 2, \ldots, m)$, and obtain the modal equations of motion

$$\ddot{q}_r(t) + \omega_r^2 q_r(t) = Q_r(t), \quad Q_r(t) = -\sum_{s=1}^m \left( g_{rs} \dot{q}_s(t) + h_{rs} q_s(t) \right) \quad (r = 1, 2, \ldots, m) \quad (5a, b)$$

where $\omega_r$ denotes the $r^{th}$ natural frequency of oscillation, $\delta_{ss}$ is the kronecker-delta function, and where from Eqs. (2) and (3) $Q_r(t)$ denote modal control forces in which

$$g_{rs} = \sum_{e=1}^n \phi_r(x_{se}) \phi_s(x_e) g_{e}, \quad h_{rs} = \sum_{e=1}^n \phi_r(x_{se}) \phi_s(x_e) h_{e} \quad (r, s = 1, 2, \ldots, m) \quad (6)$$

denote modal displacement gains and modal velocity gains, respectively. Substituting Eqs. (5b) and (6) into (5a) yields the closed-loop modal equations of motion, written compactly in the matrix-vector form

$$\ddot{q} + H \dot{q} + (G + \Lambda) q = 0 \quad (7)$$

where $q = (q_1, q_2, \ldots, q_m)^T$ denotes the modal displacement vector, $h_{rs}$ are the entries of the $H$, $g_{rs}$ are the entries of $G$ and $\omega_r^2 \delta_{ss}$ are the entries of $\Lambda$. In the absence of control, $G = H = 0$ and $\Lambda$ is diagonal, in which case the modal equations are uncoupled. In the presence of control, $G$ and $H$ are in general non-diagonal resulting in a coupled set of modal equations. The net effect is that the natural modes of vibration cease to be modes of vibration for the system in the presence of feedback control. However, when Node Control is employed, the modal control gain matrices $G$ and $H$ are diagonal implying that the natural modes of vibration of the system remain modes of vibration for the controlled system.

When the number of modes $m$ is taken to be the same as the number of actuator-sensor pairs $n$, the displacement gains $g_e$ ($r = 1, 2, \ldots, n$) and the velocity gains $h_e$ ($r = 1, 2, \ldots, n$) are given by

$$g_e = \alpha^2 \left[ \sum_{r=1}^n \phi_r^2(x_e) \right]^{-1}, \quad h_e = 2\alpha \left[ \sum_{r=1}^n \phi_r^2(x_e) \right]^{-1} \quad (e = 1, 2, \ldots, n) \quad (8)$$
in which \(\alpha\) denotes the desired exponential decay rate of the motion. The entries of \(G\) and \(H\) are 
\[ g_{rs} = \alpha^2 \delta_{rs} \] 
and 
\[ h_{rs} = 2\alpha \delta_{rs} (r = 1, 2, \ldots, n), \] respectively [5].

**Uniform Spacing**

The central question of this note concerns the requirements associated with colocation. Indeed, how closely must an actuator and sensor be spaced before an actuator-sensor pair can rightfully be referred to as colocated. Consider first the situation in which the sensors are uniformly spaced an amount \(\delta x\) from the actuators so that 
\[ x_e = x_{oe} + \delta x \quad (e = 1, 2, \ldots, n). \] 
Figure 1 shows \(h_{nn}/2\alpha\) versus \(\delta x\) of a pinned-pinned beam for \(n = 2, 3, 4, \) and \(5,\) in which \(n\) represents the number of actuator-sensor pairs, taken equal to the number of modes assumed to participate in the overall response of the system; the higher modes (greater than \(n\)) being neglected. The \(n = 1\) case was not considered because the beam remains stable regardless of \(\delta x\) in this case. In the cases shown, the instability occurred in the highest mode participating in the response, that is the \(n^{th}\) mode. The instability occurred when \(h_{nn}\) changed sign from positive to negative. Observe that the rate of decrease of \(h_{nn}\) with respect to \(\delta x\) increases with \(n.\) Figure 2 shows \(h_{nn}/2\alpha\) versus \(\delta x\) of a cantilevered beam for \(n = 2, 3, 4\) and \(5.\) As before, the \(n = 1\) case was not considered. Again, the instability consistently occurred in the highest mode and the rate of decrease of \(h_{nn}\) with respect to \(\delta x\) increases with \(n.\)

Next, we examine the critical level of uniform spacing which is defined as the greatest distance between the actuators and sensors without causing an instability. In the case of a pinned-pinned beam, the critical level of uniform spacing can be determined analytically. From Eq. (6)

\[ h_{nn} = \sum_{n=1}^{n} \phi_n(x_{oe})\phi_n(x_{oe} + \delta x)h_e = \frac{2h}{mL} \sum_{n=1}^{n} \sin \frac{m\pi x_{oe}}{L} \sin \frac{m(x_{oe} + \delta x)}{L} \]

(9)

since the \(n^{th}\) natural mode of vibration of a pinned-pinned beam is 
\[ \phi_n(x) = \sqrt{\frac{2}{mL}} \sin \frac{m\pi x}{L}, \]
and the physical gains are uniform in the case of a pinned-pinned beam, that is \(h_e = h (e = 1, 2, \ldots, n).\) The critical level of uniform spacing \(\delta x_{cr}\) is determined by letting \(h_{nn} = 0\) in Eq. (10). We get
$$\delta x_\alpha = \frac{L}{m \csc \frac{\pi}{n}} \left[ \frac{\sum_{\alpha=1}^{n} \sin \frac{2\pi \alpha e}{n+1}}{\sum_{\alpha=1}^{n} \cos \frac{2\pi \alpha e}{n+1} - n} \right].$$

(10)

since \(x_\alpha = \frac{eL}{(n+1)}\) (\(e = 1, 2, \ldots, n\)). Table 1 shows \(\delta x_\alpha\) versus \(n\) and versus the \(n^{th}\) wavelength \(\lambda_n = \frac{2L}{n}\). Observe from Table 1, the rule of thumb

$$\delta x_\alpha = \frac{\lambda_n}{4}$$

(11)

in which \(n\) represents the highest mode participating significantly in the overall response of the pinned-pinned beam. Although Table 1 implies Eq. (11), it is not a proof. The proof is given elsewhere.

For beams, other than uniform pinned-pinned beam, Eq (11) is invalid. The invalidation is readily anticipated when we recognize that the wavelength of other types of beams, such as a cantilevered beam, changes as we move along the long axis of the beam. Therefore, some ambiguity arises with the reference to a wavelength within the context of beams other than uniform pinned-pinned beams. Furthermore, the determination of an analytical expression for \(\delta x_\alpha\) is difficult if not impossible for beams other than uniform pinned-pinned beams. Therefore, for beams other than uniform pinned-pinned beams we resort to numerical methods of determining \(\delta x_\alpha\), and modify our definition of wavelength.

For a cantilevered beam, the normalized modes of vibration are

$$\phi_n = \frac{1}{mL} \left[ (\sin \beta_n x - \sinh \beta_n x) \left( \frac{c + ch}{s + sh} \right) - (\cos \beta_n x - \cosh \beta_n x) \right] \quad (n = 1, 2, \ldots),$$

in which \(c \equiv \cos \beta_n L\), \(s \equiv \sin \beta_n L\), \(ch \equiv \cosh \beta_n L\), and \(sh \equiv \sinh \beta_n L\). Table 2 shows \(\delta x_\alpha\) versus \(n\) and versus the \(n^{th}\) wavelength for a uniform cantilevered beam. The wavelength \(\lambda_n\) was defined as twice the distance between the two right most nodes of the \(n^{th}\) mode. Referring to Table 2, the rule
of thumb given by Eq. (11) holds with a satisfactory level of accuracy, although it is not exact. Also observe that we took negative spacing because the beam is cantilevered.

In Tables 1 and 2 the number of actuator-sensor pairs of a pinned-pinned beam was taken to be equal to the number of modes participating in the response. Table 3 treats cases in which the number \( n \) of actuator-sensor pairs of a pinned-pinned beam is different than the number \( m \) of modes participating in the system response. Observe in Table 3 that the values of \( \delta x_{cr} \) in a single row are all equal to each other except for those values that are infinity. The reason for these infinities is connected to a property of Node Control - that the actuators are placed at the nodes of the \( n+1^{\text{st}} \) mode, further implying that the actuators are not capable of controlling the \( n+1^{\text{st}} \) mode and its multiples. For example, observe that modes 3, 6 and 9 of a pinned-pinned beam are unaffected when two actuators are placed at the nodes of mode 3.

Furthermore, observe that \( \delta x_{cr} \) does not depend on the number of actuator-sensor pairs \( n \). Instead \( \delta x_{cr} \) depends on the number of modes \( m \) participating in the response. This is also proven elsewhere in the case of a uniform pinned-pinned beam\(^8\). In the view of this, the rule of thumb, Eq.(11), needs to be modified to

\[
\delta x_{cr} = \frac{\lambda_m}{4} \tag{12}
\]

in which \( m \) denotes the number of modes participating in the overall response of the beam.

Equation (12) states that the critical level of uniform spacing of a uniform pinned-pinned beam is a quarter wavelength for all of the modes except for the multiples of the \( n+1^{\text{st}} \) mode. For the modes that are multiples of the \( n+1^{\text{st}} \) mode, the critical level of uniform spacing is infinite, i.e., those modes tolerate an infinite amount of non-colocation, at least in theory.

**Non-uniform Spacing**

Consider now the situations in which sensors are spaced nonuniformly away from the actuators. Specifically, we let the sensors be uniformly spaced from their actuators except for one sensor. We substitute the \( \phi_i(x) \) of a pinned-pinned beam into Eq.(6) like we did for uniform
spacing, but this time without letting $\delta x_e = \delta x$ for all $e$. We let $\delta x_e = \delta x$ for $e \neq t$ and $\delta x_t \neq \delta x$ and after some manipulation get
\[
\frac{mL}{2h} h_{rr} + \sin \frac{r \pi x_r}{L} \sin \frac{r \pi (x_r + \delta x)}{L} = \sum_{e \neq t} \sin \frac{r \pi x_e}{L} \sin \frac{r \pi (x_e + \delta x_e)}{L} + \sin \frac{r \pi x_t}{L} \sin \frac{r \pi (x_t + \delta x_t)}{L} \quad (13)
\]

Letting $\delta x = \delta x_{cr}$ associated with the critical level of uniform spacing obtained earlier, we get
\[
\sum_{e \neq t} \sin \frac{r \pi x_e}{L} \sin \frac{r \pi (x_e + \delta x_e)}{L} = 0 , \text{ so}
\]
\[
h_{rr} = \frac{2h}{mL} \sin \frac{r \pi x_t}{L} \left( \sin \frac{r \pi (x_t + \delta x_t)}{L} - \sin \frac{r \pi (x_t + \delta x_t)}{L} \right) \quad (14)
\]

Figures 3(a)-3(d) shows $h_{nm}$ versus $\delta x_t$ (letting $t = 1$ in Eq (14)). Observe that the first positive intersection of each curve with the x-axis is identical to $\delta x_{cr}$. For example, the first positive intersection in Fig. 3(d) is 0.1, which is identical to $\delta x_{cr}$ associated with five participating modes ($\frac{\lambda x}{4} = \frac{12L}{4} = 0.1L$). Also observe in Figs. 3(a)-3(d) that $h_{nn}$ is positive (stable) at the left side of the intersection and negative (unstable) at the right. Therefore moving the first sensor closer to the actuator stabilizes the system while moving it further away from the actuator causes the system to become unstable.

Fig. 4(a)-4(i) show how moving ten different sensors on a pinned-pinned beam effects the stability of the 10th mode. Again, we can see if we move one of the these sensors away from $\delta x_{cr}$ (0.05L), that the change in stability depends on which direction the sensor is moved. Moving the sensor away from its actuator creates an instability while moving it closer to the actuator has a stabilizing effect. However if we compare the curves in Fig. 4, we find that the intervals within which $h_{nn}$ is positive become smaller as we turn from Fig. 4(a), which moves the first of the ten sensors, to Fig 4(i), which moves the tenth of the ten sensors. Figures 4(h) and 4(i) also show that if all of the sensors are non-colocated at $\delta x_{cr}$ except the ninth or tenth ones, the system is unstable. It follows that it is important to know which sensor on the beam is colocated and which one is not.
Conclusions

This note addressed the question as to the requirements associated with actuator-sensor colocation in the vibration suppression of structures. As a starting point, a direct feedback control approach was considered. Node Control, which is a simple method of placing actuators, and assigning direct feedback control gains in vibration suppression of structures, was selected as the method of control. Realizing that the number of possible non-colocated cases is infinite, we broke the problem down into first uniform spacing, and then uniform spacing except for one actuator-sensor pair. We looked at two systems - a pinned-pinned beam and a cantilevered beam. The results associated with the pinned-pinned beam were handled analytically and those for the cantilevered beam were handled numerically.

In the case of uniform spacing, we found that the critical level of uniform spacing was identical to a quarter wavelength of the highest mode participating in the response. This result was independent of the number of actuator-sensors pairs. As shown in Tables 1, 2 and 3, we found that

\[ \delta x_{cr} = \frac{\lambda_m}{4} \]

In the non-uniform spacing case, we found that spacing a single sensor slightly less than the critical amount (while the others remained spaced the critical amount) results in an increase in stability. On the other hand, increasing the spacing of a single sensor or significantly decreasing the spacing of a single sensor generally caused instabilities. This is shown in Figs. 3 and 4.

The results described in this note have implications in structures problems where the geometry and boundary conditions are more complex, e.g., in one-dimensional structures and in two-dimensional structures like plate and shell-like structures. It is expected, that the results obtained here would be similar to the results obtained for those structures. Of course, there is no guarantee.
References


Table 1. Critical level of uniform spacing of a pinned-pinned beam

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<th>$n$</th>
<th>$\delta x_{cr}$</th>
<th>$\frac{\Lambda_n}{4}$</th>
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<tr>
<td>1</td>
<td>0.5000L</td>
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Table 2. Critical level of uniform spacing of a cantilever beam

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<tr>
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Table 3: ER of a pinned-pinned beam controlled by n-actuator-sensor pairs assuming m modes participating in the response.
Figure captions

Figure 1. Uniform spacing $\delta x_e = \delta x$ ($e = 1, 2, ..., n$); $\frac{h_{en}}{2\alpha}$ versus $\delta x$ for a pinned-pinned beam

Figure 2. Uniform spacing $\delta x_e = \delta x$ ($e = 1, 2, ..., n$); $\frac{h_{en}}{2\alpha}$ versus $\delta x$ for a cantilevered beam.

Figure 3. Nonuniform spacing $\delta x_e = \delta x_e$ ($e = 1, 2, ..., n$); $\frac{h_{en}}{2\alpha}$ versus $\delta x_1$ for a pinned-pinned beam

Figure 4. Nonuniform spacing $\delta x_e = \delta x_e$ ($e = 1, 2, ..., n$) except for $\delta x_e$; $\frac{h_{1\alpha}}{2\alpha}$ versus $\delta x_e$ for a pinned-pinned beam