simple GA is slow to evolve but after 120 generations (18,000 function evaluations) reaches a near-optimal solution. The boundary conditions for the best solution in the simple GA are $r(t_f) = 1.519 DU$, $\theta(t_f) = 153.41^\circ$ DU, $u(t_f) = 0.051 DU/TU$, and $v(t_f) = 0.807 DU/TU$ and for the $\mu$GA with a population of 15 are $r(t_f) = 1.514 DU$, $\theta(t_f) = 162.59^\circ$, $u(t_f) = 0.206 DU/TU$, and $v(t_f) = 0.805 DU/TU$. The $\mu$GAs' radial position and tangential velocity are close to the desired values given in Eq. (14) but the radial velocity would make the final orbit eccentric.

Results: Inequality Constraints Method

Figure 1 also displays the objective function for the best individual in the population using inequality constraints. The vertical scale on the right should be used. It is seen that, given a sufficient number of generations, all three GAs converged to a near-optimal solution. However, the $\mu$GAs converged to the near-optimal region more quickly than did the simple GA. The performance of all GAs became comparable after approximately 7500 function evaluations. The boundary conditions for best solution in the GA with a population of 15 after 5000 are $r(t_f) = 1.509 DU$, $\theta(t_f) = 153.41^\circ$, $u(t_f) = 0.042 DU/TU$, and $v(t_f) = 0.808 DU/TU$ and after 12,000 function evaluations are $r(t_f) = 1.524 DU$, $\theta(t_f) = 152.59^\circ$, $u(t_f) = 0.051 DU/TU$, and $v(t_f) = 0.807 DU/TU$.

Orbital rendezvous trajectories have also been solved using $\mu$GAs with inequality constraints but are not shown here. In all cases the $\mu$GA provides an extremely fast approach to a near-optimal trajectory.

Conclusions

The use of $\mu$GAs to determine near-optimal low-thrust trajectories was explored. Micro-GAs were inefficient at achieving near-optimal solutions when boundary conditions were treated as equality constraints. However, when boundary conditions were cast as inequality constraints, $\mu$GAs showed faster convergence than did simple GAs to a near-optimal region.

Acknowledgment

The author would like to acknowledge David Carroll for providing the GA driver used in this study and for valuable discussions pertaining to its use.

References


Conjecture About Orthogonal Functions

Larry Silverberg* North Carolina State University, Raleigh, North Carolina 27695-7910

Introduction

I FIRST came upon this result in my work in the area of control of distributed systems. I was studying the manner in which the natural modes of vibration of simple beams are altered by attaching to them concentrated spring and damping elements. The result has since been distilled into an unproved theorem, presently being called the orthogonal function conjecture.

Orthogonal function conjecture: Let $\phi_n(x)$ be an ordered set of real orthonormal functions defined over the interval $[a, b]$ of the $(n+1)$th orthogonal functions, are $x_1, x_2, \ldots, x_n$. Then an orthonormal set of real orthonormal vectors can be constructed from the orthonormal functions by evaluating the lowest $n$ orthonormal functions at the zeros of the $(n+1)$th orthogonal function. The orthonormal vectors are $\psi_j = [w_1\phi_1(x_1) \quad w_2\phi_2(x_2) \quad \ldots \quad w_n\phi_n(x_n)]^T$ in which $w_j = (r = 1, 2, \ldots, n)$ are positive numbers.

This described orthogonal function conjecture is both unusual and a paradox. It is unusual because the zeros of the $(n+1)$th orthonormal function influence the construction of orthonormal vectors from the lowest $n$ orthonormal functions. The orthogonal function conjecture is a paradox for reasons described in the next section.

Paradox

Let us now more closely examine the orthogonal function conjecture. The orthonormality conditions that the functions $\phi_n(x)$ satisfy are given by

$$\int_0^1 \phi_n(x) \phi_m(x) \, dx = \delta_{nm}, \quad (r = 1, 2, \ldots, n + 1) \quad (1)$$

where $\delta_{nm}$ is the Kronecker delta function ($\delta_{nm} = 0$ when $n \neq m$ and $\delta_{nm} = 1$) and $x$ is defined over the interval $[0, 1]$. The zeros of $\phi_n(x)$ satisfy

$$\phi_{n+1}(x_t) = 0 \quad (t = 1, 2, \ldots, n) \quad (2)$$

It is implied by Eq. (2) that $\phi_{n+1}(x_t)$ has $n$ zeros. The $n$-dimensional orthonormal vectors are stated in the conjecture to satisfy the orthonormality conditions

$$\psi^T \psi = \delta_{rr}, \quad (r = 1, 2, \ldots, n) \quad (3)$$

where $\psi_j = [w_1\phi_1(x_1) \quad w_2\phi_2(x_2) \quad \ldots \quad w_n\phi_n(x_n)]^T$ in which $w_j$ shall be referred to as weighting constants. The paradox arises when we recognize that Eq. (3) represents a set of linear algebraic equations in terms of the unknowns $w_j = (r = 1, 2, \ldots, n)$. The number of equations is equal to $n^2$, and the equation corresponding to the pair of indices $(r, s)$ is identical to the equation corresponding to the pair of indices $(s, r)$, therefore, the number of independent equations is $N = n(n + 1)/2$. The paradox lies in that the number of equations.
is larger than the number of unknowns (when \( n > 1 \)). Indeed, the existence of an exact solution to an overdetermined set of linear algebraic equations is altogether unexpected. Without an available explanation, the presence of an exact solution must be regarded as a paradox.

**Sine Functions**

Without a proof of the orthogonal function conjecture, I now resort to verifying the conjecture. The verification is given to provide insight that could lead to a proof of the conjecture at a later date. The verification could also illuminate restrictive conditions to which the conjecture is subject. First consider the orthonormal set of sine functions

\[
\phi_r(x) = \sqrt{2} \sin r\pi x \quad (r = 1, 2, \ldots, n + 1)
\]

(4)

The zeros of \( \phi_{n+1}(x) \) are

\[
x_i = t/(n + 1) \quad (r = 1, 2, \ldots, n)
\]

(5)

The weights \( w_r \) (\( r = 1, 2, \ldots, n \)) associated with the orthonormal vectors \( \phi_r \) (\( r = 1, 2, \ldots, n \)) are obtained by letting \( r = s \) in Eq. (3). This yields the set of \( n \) linear algebraic equations

\[
\sum_{i=1}^{n} \phi_i^2(x_i) w_i^2 = 1 \quad (r = 1, 2, \ldots, n)
\]

(6)

Letting \( r \neq s \) in Eq. (3) yields the unproven identities

\[
\sum_{i=1}^{n} \phi_i(x_i) \phi_s(x_i) w_i^2 = 0 \quad (r \neq s = 1, 2, \ldots, n)
\]

(7)

Substituting Eqs. (4) and (5) into Eqs. (6) and (7), we get

\[
\sum_{i=1}^{n} 2 \sin^2 \left( \frac{r \pi t}{n + 1} \right) w_i^2 = 1 \quad (r = 1, 2, \ldots, n)
\]

(8)

and

\[
\sum_{i=1}^{n} \sin \left( \frac{r \pi t}{n + 1} \right) \sin \left( \frac{s \pi t}{n + 1} \right) w_i^2 = 0 \quad (r \neq s = 1, 2, \ldots, n)
\]

(9)

As an illustration, let \( n = 3 \), in which case Eqs. (8) and (9) reduce to

\[
\begin{bmatrix}
1 & 2 & 1 \\
2 & 0 & 2 \\
1 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
w_1^2 \\
w_2^2 \\
w_3^2
\end{bmatrix}
= \begin{bmatrix} 1 \\
1 \\
1
\end{bmatrix}
\]

(10a)

\[
\begin{bmatrix}
\sqrt{2} & 0 & -\sqrt{2} \\
1 & -2 & 1 \\
\sqrt{2} & 0 & -\sqrt{2}
\end{bmatrix}
\begin{bmatrix}
w_1^2 \\
w_2^2 \\
w_3^2
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix}
\]

(10b)

Observe that the first and third equations in Eq. (10a) are identical to each other. The general solution to Eq. (10a) is then expressed in terms of \( w_1^2 \) as \( w_2^2 = w_1^2 \frac{1}{2} - w_1^2 \). Substituting the general solution to Eq. (10a) into Eq. (10b) verifies Eq. (10b) when \( w_1 = \frac{1}{2} \). The verification of Eq. (9) for values of \( n \) ranging from 1 to 10 was carried out elsewhere.\(^1\)

At this point in the verification, one might speculate that the satisfaction of Eq. (9) is peculiar to the sine functions given in Eq. (4). Let us look at another orthonormal set of sine functions, given by

\[
\phi_r(x) = \sqrt{2} \sin r\pi x \quad (r = 1, 2, \ldots, n + 1)
\]

(11a)

\[\beta \cot \beta + \gamma = 0 \quad (r = 1, 2, \ldots, n + 1)
\]

(11b)

in which \( \gamma > 0 \) (Ref. 3). The zeros of \( \phi_{n+1}(x) \) satisfy

\[
x_i = \frac{\pi}{2\beta_{n+1}} (2t - 1) \quad (t = 1, 2, \ldots, n)
\]

(12)

in which \( \beta_{n+1} \) is obtained from Eq. (11b). Substituting Eqs. (11a) and (12) into Eqs. (6) and (7) yields

\[
\sum_{i=1}^{n} 2 \left( \frac{\beta_i^2 + \gamma^2}{\beta_i^2 + \gamma^2 + \gamma} \right) \sin \left[ \frac{\beta_i}{\beta_{n+1}} \frac{\pi}{2} (2t - 1) \right] w_i^2 = 1
\]

\[
(r = 1, 2, \ldots, n)
\]

(13)

and

\[
\sum_{i=1}^{n} \left[ \frac{4 (\beta_i^2 + \gamma^2)}{(\beta_i^2 + \gamma^2 + \gamma)} \frac{\beta_i}{\beta_{n+1}} \frac{\pi}{2} (2t - 1) \right] w_i^2 = 0
\]

\[
(r \neq s = 1, 2, \ldots, n)
\]

(14)

As an illustration, let \( n = 5 \) and \( \gamma = 2 \). From Eq. (11b), \( \beta_1 = 2.2889, \beta_2 = 5.0870, \beta_3 = 8.0962, \beta_4 = 11.7277, \beta_5 = 14.2764, \) and \( \beta_6 = 17.3932 \). From Eq. (12), the zeros of \( \phi_r \) are given by \( x_1 = 0.1806, x_2 = 0.3612, x_3 = 0.5419, x_4 = 0.7225, \) and \( x_5 = 0.9031 \). By inversion of Eq. (13), the weighting constants \( w_1 = 0.5158, w_2 = 0.4950, w_3 = 0.4700, w_4 = 0.4433, \) and \( w_5 = 0.4144 \) are found. Substituting these values into the left-hand side of the unproven identities (14) completes the verification of the orthogonal function conjecture for the sine functions given by Eq. (11a) restricted to \( n = 5 \). The verification of Eq. (14) for values of \( n \) ranging from 1 to 15 was carried out elsewhere.\(^3\)

The verification just given considered two orthonormal sets of sine functions. The sets of orthonormal functions for which the orthogonal function conjecture has been verified are given in Table I. The orthogonal function conjecture was violated when the set of orthonormal functions was associated with free-free beams.\(^1\) This set is a mixed set of two polynomial functions, and the remaining are transcendental functions. Observe in Table I that the orthogonal function conjecture was satisfied for another mixed set (case 4).

**Isolated Proof for the Functions**

\[\sqrt{2} \sin (r \pi x) \quad (r = 1, 2, \ldots, n + 1)\]

Although the author has presently failed to prove the orthogonal function conjecture with any level of generality, the set of functions \( \sqrt{2} \sin (r \pi x) \) \((r = 1, 2, \ldots, n + 1)\) is an isolated exception.

Toward this end, an attractive method of computing the weighting constants \( w_r \) \((r = 1, 2, \ldots, n)\) is first developed. The method avoids the matrix inverse that is required in Eq. (6). Adapting a matrix notation, and letting \( \phi_r = \phi_r(x) \) \((r = 1, 2, \ldots, n)\) denote the entries of \( \Phi \) and \( w_r = w_r \delta_r \) \((r = 1, 2, \ldots, n)\) denote the entries of \( W \), the matrix counterpart to Eq. (3) becomes

\[
\psi^T \psi = I
\]

(15a)

in which

\[
\psi = W \Phi
\]

(15b)

where \( I \) is the \( n \times n \) identity matrix. Substituting Eq. (15b) into Eq. (15a) yields \( \Phi^T W^2 \Phi = I \) since \( W^T W = W^2 \). Premultiplying this result by \( \Phi^{-T} \), and postmultiplying this result by \( \Phi^{-1} \) (assuming that \( \Phi^{-1} \) exists), leads to \( W^2 = \Phi^{-T} \Phi^{-1} = (\Phi \Phi^T)^{-1} \), which upon inversion yields

\[
W^2 = \Phi \Phi^T
\]

(16)

in which the entries of \( W^{-2} \) are \((1/w_r^2) \delta_{rs} \). Returning to an index notation, Eq. (16) becomes

\[
\left( \frac{1}{w_r^2} \right) \delta_{rs} = \sum_{i=1}^{n} \phi_i(x_r)\phi_i(x_s) \quad (r, s = 1, 2, \ldots, n)
\]

(17)

When \( r = s \), Eq. (17) is an attractive equation to use to determine \( w_r \) \((r = 1, 2, \ldots, n)\). When \( r \neq s \), Eq. (17) represents an alternate form of the unproven identities (7). Thus, from now on a proof of Eq. (17)
Table 1 Verification of orthogonal function conjecture

<table>
<thead>
<tr>
<th>Transcendental functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_r(x) = \sqrt{2} \sin \left[ \frac{(2r - 1)\pi x}{2} \right] ), ( 0 &lt; x &lt; 1 ) ( (r = 1, 2, \ldots, n + 1; n = 1, 2, \ldots, 10) )</td>
</tr>
<tr>
<td>( \phi_r(x) = \sqrt{2} \sin \frac{\beta r^2 + y^2}{\beta^2 + y^2 + y} ), ( \beta r \cos \beta r + \gamma = 0 ), ( \gamma = 2 ), ( 0 &lt; x &lt; 1 )</td>
</tr>
<tr>
<td>( \phi_0(x) = 1 ), ( \phi_r(x) = \sqrt{2} \cos \left( \pi r x \right) ), ( 0 &lt; x &lt; 1 ) ( (r = 1, 2, \ldots, n + 1; n = 1, 2, \ldots, 10) )</td>
</tr>
<tr>
<td>( \phi_r(x) = \sqrt{2} \beta r^2 + y^2 + y ) ( \cos \beta r, x, 0 &lt; x &lt; 1 ), ( \beta r \tan \beta r = \gamma ), ( \gamma = \frac{1}{3} )</td>
</tr>
<tr>
<td>( \phi_r(x) = 1 ), ( \phi_2(x) = 10(x^4 + 3x^2/14 + 3/560) ), ( 0 &lt; x &lt; 1 ) ( (n = 1, 2, \ldots, 10) )</td>
</tr>
</tbody>
</table>

is sought when \( r \neq s \), restricting our attention to the orthonormal set of functions \( \sqrt{2} \sin (r \pi x) \) \( (r = 1, 2, \ldots, n + 1) \).

Within our restrictive context, Eq. (17) for \( r \neq s \) reduces to:

\[
0 = \sum_{t=1}^{n} \sin \frac{r \pi t}{n+1} \sin \frac{s \pi t}{n+1} = \sum_{t=1}^{n} \cos \frac{(r-s)\pi t}{n+1} \cos \frac{(r+s)\pi t}{n+1} \quad (r \neq s = 1, 2, \ldots, n) \tag{18}
\]

When \( r \pm s = 2k \) is even, the identity

\[
\sum_{t=1}^{n} \cos \frac{2\pi kt}{n+1} = 0 \quad (k = 1, 2, \ldots, n)
\]

can be used to show that

\[
\sum_{t=1}^{n} \cos \frac{(r \pm s)\pi t}{n+1} = -1
\]

It follows that Eq. (18) is satisfied when \( r \pm s \) is even. It remains to prove Eq. (18) when \( r \pm s = k \) is odd. First, consider the even \( n \) cases, for which

\[
\sum_{t=0}^{n+1} \cos \frac{\pi kt}{n+1} = \sum_{t=0}^{n/2} \cos \frac{\pi kt}{n+1} + \sum_{t=(n+1)/2}^{n+1} \cos \frac{\pi kt}{n+1} = \sum_{t=0}^{n/2} \cos \frac{\pi kt}{n+1} - \sum_{t=n/2}^{n-1} \cos \frac{\pi k(t'-n/2)}{n+1}
\]

letting \( t' = n + 1 - t \). Then

\[
\sum_{t=0}^{n+1} \cos \frac{\pi kt}{n+1} = \sum_{t=0}^{n/2} \cos \frac{\pi kt}{n+1} - \sum_{t=n/2}^{n-1} \cos \frac{\pi k(t'-n/2)}{n+1} = 0
\]

and so

\[
\sum_{t=1}^{n} \cos \frac{\pi kt}{n+1} = \sum_{t=1}^{n+1} \cos \frac{\pi kt}{n+1} - \cos(0) - \cos(\pi k) = 0
\]

for odd \( k \). Finally, let us prove Eq. (18) for odd \( r \pm s = k \) when \( n \) is odd. Under these conditions

\[
\sum_{t=0}^{n+1} \cos \frac{\pi kt}{n+1} = \sum_{t=0}^{(n+1)/2} \cos \frac{\pi kt}{n+1} + \cos \frac{\pi k(n+1)/2}{n+1} - \sum_{t=(n+1)/2}^{n+1} \cos \frac{-\pi kt}{n+1} + \pi k
\]

\[
= \sum_{t=0}^{(n+1)/2} \cos \frac{\pi kt}{n+1} - \sum_{t=(n+1)/2}^{n+1} \cos \frac{\pi k(t'-n/2)}{n+1} = 0
\]

letting \( t' = n + 1 - t \). Then

\[
\sum_{t=0}^{n+1} \cos \frac{\pi kt}{n+1} = \sum_{t=0}^{(n+1)/2} \cos \frac{\pi kt}{n+1} - \sum_{t=(n+1)/2}^{n+1} \cos \frac{\pi k(t'-n/2)}{n+1} = 0
\]

so that for odd \( k \)

\[
\sum_{t=1}^{n} \cos \frac{\pi kt}{n+1} = \sum_{t=1}^{n+1} \cos \frac{\pi kt}{n+1} - \cos(0) - \cos(\pi k) = 0
\]

End of proof.

Node Control Conjecture

This section is for readers interested in the application of the orthogonal function conjecture; readers uninterested in this topic may wish to skip this section. An application of the orthogonal function conjecture to the control of distributed systems is now restated in the form of the node control conjecture. The node control conjecture was first presented in Ref. 1 and is presented again here to show how the orthogonal function conjecture can be used to determine control gains and to determine control input locations in direct state feedback control of one-dimensional systems. (The node control conjecture has been used to control transient temperatures in conducting media in Ref. 3 and to control vibration in two-dimensional plates and membranes in unpublished work by the authors of Ref. 2.)
Consider the partial differential equation that governs the transient vibration of a one-dimensional uniform bar

\[ m \frac{\partial^2 u(x,t)}{\partial t^2} - AE \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t) \]  \hspace{1cm} (19)

where \( u(x,t) \) is a longitudinal displacement at \( 0 < x < 1 \) and time \( t > 0 \), \( f(x,t) \) is the control force distribution, \( m \) is the mass distribution, and \( AE \) is the stiffness distribution. Both \( m \) and \( AE \) are positive constants. The uniform bar is subject to the boundary conditions

\[ -k_1 u(0,t) + \frac{\partial u(0,t)}{\partial x} = 0, \quad k_2 u(1,t) + \frac{\partial u(1,t)}{\partial x} = 0 \]  \hspace{1cm} (20)

in which \( k_1 \) and \( k_2 \) are nonnegative constants. The control force distribution is

\[ f(x,t) = -\sum_{i=1}^{n} \left[ g_i u(x,t) + h_i \frac{\partial u(x,t)}{\partial t} \right] \delta(x-x_i) \]  \hspace{1cm} (21)

in which \( \delta(x-x_i) \) is a spatial impulse at the undetermined location \( x_i \) and \( g_i \) and \( h_i \) are undetermined control gains. The eigenfunctions of the uniform bar solve the eigenvalue problem

\[ \lambda^2 m \phi(x) - AE \frac{\partial^2 \phi(x)}{\partial x^2} = -\sum_{i=1}^{n} (g_i + h_i \lambda) \phi(x) \delta(x-x_i) \]  \hspace{1cm} (22)

subject to the homogeneous boundary conditions

\[ -k_1 \phi(0) + \frac{\partial \phi(0)}{\partial x} = 0, \quad k_2 \phi(1) + \frac{\partial \phi(1)}{\partial x} = 0 \]

There exists a countably infinite number of eigenfunctions \( \phi_i(x) \) \( (r = 1, 2, \ldots) \) and associated eigenvalues \( \lambda_r \), that solve Eq. (18). In the absence of feedback control [letting \( g_i = h_i = 0 \) \( (r = 1, 2, \ldots, n) \) in Eq. (22)] the eigensolutions are

\[ \phi^0_i(x) = N_i \left[ (k_1 - \beta^2) \sin \beta x + \beta \cos \beta x \right] \]

\[ \lambda^0_r = \sqrt{\frac{\alpha}{m}} \left( k_1 + 2 k_2 \beta \right) \]  \hspace{1cm} (23)

in which \( \beta \) satisfy the characteristic equation

\[ (k_1 - k_2 - \beta^2) \tan \beta + (k_1 + k_2) \beta = 0 \]

and the superscript 0 designates quantities associated with the undetermined system. The eigenfunctions of the undetermined system are also referred to as natural modes of vibration. The eigenfunctions of the controlled system are also called controlled modes of vibration. The locations of the forces \( x_i \) \( (r = 1, 2, \ldots, n) \) and the control gains \( g_i \) and \( h_i \) \( (r = 1, 2, \ldots, n) \) are determined on the basis of the following node control conjecture.

Node control conjecture: A uniform bar with homogeneous boundary conditions in which the lowest \( n \) natural modes of vibration participate significantly in the system response, when subject to \( n \) direct state feedback forces placed at the zeros of the \( (n+1) \) th natural mode, can be controlled in a manner that satisfies the following three properties.

1) Mode invariance: the \( n \) controlled modes of vibration are identical to the natural modes of vibration.

2) Frequency invariance: the frequencies of oscillation of the \( n \) controlled modes of vibration are identical to the natural frequencies of oscillation.

3) Uniform damping: the damping rates of the \( n \) controlled modes of vibration are identical to each other.

The desirability associated with these three properties was described in detail in Ref. 4. In short, the first two properties minimize the magnitude of the direct feedback control forces. Control forces increase in magnitude when tasked to effectively change the bar's natural modes of vibration and to change the bar's natural frequencies of oscillation. We can see that the third property is desirable when we consider the alternatives. The allowance of one decay rate to be lower than the others yields a response that in time is dominated by that mode. Furthermore, the magnitude of a control force increases with the decay rate. It follows that the control forces will be unnecessarily large when one decay rate is either smaller or larger than the rest. The level of uniform damping is then selected on the basis of how much overall effort associated with the control forces the designer is willing to expend.

It is now shown how the node control conjecture follows from the orthogonal function conjecture. First express the \( n \) lowest controlled modes in terms of the lowest \( n \) natural modes as

\[ \phi_r(x) = \sum_{i=1}^{n} \phi^0_i(x) c_{ri} \]  \hspace{1cm} (24)

in which \( c_{rs} \) \( (r, s = 1, 2, \ldots, n) \) are called coupling coefficients. Equation (24) is tantamount to assuming that \( n \) modes participate in the system response and that the participation of the remaining modes is negligible. Substituting Eq. (24) into Eq. (22), premultiplying by

\[ \frac{1}{m} \int_0^1 \phi^0_r(x) \phi^0_s(x) \ dx \]

and applying the orthonormality conditions (1) yields the eigenvalue problem

\[ 0 = \sum_{i=1}^{n} \left[ \lambda^2 \delta_{rs} + \lambda \sum_{i=1}^{n} h_i \phi^0_i(x) \phi^0_r(x) \right] + \lambda^0 \delta_{rs} \]

\[ + \frac{1}{m} \left[ \sum_{i=1}^{n} g_i \phi^0_i(x) \phi^0_r(x) \right] c_{rs} \]  \hspace{1cm} (25)

It follows from the orthogonal function conjecture that the parenthetic summations in Eq. (25) can be reduced to the form

\[ \sum_{i=1}^{n} h_i \phi^0_i(x) \phi^0_r(x) = h \delta_{rs} \]

\[ \sum_{i=1}^{n} g_i \phi^0_i(x) \phi^0_r(x) = g \delta_{rs} \]  \hspace{1cm} (26)

Substituting Eq. (26) into Eq. (25) yields

\[ 0 = \left[ \lambda^2 + \lambda \frac{h}{m} + \lambda^0 \left( g + \frac{h}{m} \right) \right] c_{rs} = 0 \]

\[ (r, t = 1, 2, \ldots, n) \]  \hspace{1cm} (27)

Solving Eq. (27) yields

\[ c_{rs} = \delta_{rs} \]  \hspace{1cm} (r, t = 1, 2, \ldots, n) \]  \hspace{1cm} and \hspace{1cm} \[ \lambda_r = -\alpha \pm i \omega_r \]  \hspace{1cm} (28)

in which \( \alpha = (h/2m) \) is the uniform decay rate and \( \omega_r = \sqrt{(-\lambda_r)} \) is the \( r \) th natural frequency of oscillation. Indeed, Eq. (28) is a statement of the three properties predicted by the node control conjecture.

Summary

This Note introduced an unproven conjecture about orthogonal functions. The conjecture leads to a paradox that is currently unresolved except in case of the orthogonal functions \( \phi(x) = \sqrt{2} \sin \pi x \) \( (r = 1, 2, \ldots) \). The application of the conjecture to the control of distributed systems was also presented in the form of a node control conjecture.

References
