AN ENGINEERING FOUNDATION FOR CONTROLLING HEAT TRANSFER IN ONE-DIMENSIONAL TRANSIENT HEAT CONDUCTION PROBLEMS

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ABSTRACT  
This paper develops an engineering foundation for controlling heat transfer in one dimensional transient heat conduction problems based upon concepts borrowed from vibration control problems. The foundation distinguishes between modal control, distributed control, discrete control and direct feedback control and then singles out direct feedback control because its simplicity. An example demonstrates modal control and direct feedback control of the variation of the transient temperature field in a one dimensional slab geometry.

Introduction  
This paper is concerned with the development of an engineering foundation for solving control problems involving transient heat conduction. The method developed here is analogous to a method currently used for solving control problems involving structural vibration.

Before proceeding, let us briefly review the development of these engineering methods. We begin with the landmark paper entitled Control of Self-Adjoint Distributed-Parameter Systems by L. Meirovitch and Baruh [1]. In this paper, Meirovitch succinctly describes the fundamental principle behind modal control whereby a system is controlled by controlling its natural modes of vibration. Meirovitch suggests that a mechanical system's eigenfunctions are more than merely an orthogonal set of functions. More so, they belong to a particular mechanical system and in this regard are natural to that system. Thus, controlling a system is tantamount to controlling its natural modes of vibration. During the same period, M. Balas [2] developed the dual principles of control spillover and observation spillover for distributed systems controlled by discrete inputs. It was shown that discrete inputs acting on a distributed system necessitates the distinction between controlled modes and residual modes. He showed that the residual...
modes are unavoidably excited by the discrete inputs. Balas named the excitement of the residual modes control spillover\(^1\) and, later on, in dual theories, the terms observation spillover and identification spillover were coined [2,3].

The two works by Meirovitch and Balas led to a flourish of about 500 papers. The topics of these papers can be categorized into several areas including modal control, distributed control, discrete control and direct feedback control; their main distinguishing features and interrelationships as applied to heat conduction will be brought out in this paper.

Section two reviews the modal equations of the one dimensional transient heat conduction problem. The restriction to a one dimensional domain is made mostly to simplify the exposition. The exception to this is the criteria upon which we base the location of the discrete control inputs\(^2\) which can be generalized only in special cases to two and three dimensional heat conduction problems. Otherwise, the principles developed in this paper can be extended to problems in two and three dimensional domains. The third section discusses modal control and within this development distinguishes between distributed control and discrete control. It will be shown that modal control is capable of independently controlling the damping rate of each mode of temperature. The fourth section develops direct feedback control as a byproduct of modal control, where it is shown that direct feedback can uniformly dampen the controlled modes of temperature; a desirable result in engineering problems. Section five presents an example illustrating the application of modal control and direct feedback control. Finally section six presents a summary of this work.

The Modal Equations of One-Dimensional Heat Conduction

We begin with the one-dimensional heat conduction problem defined by the equation

\[
\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + g = \rho C_v \frac{\partial T}{\partial t}, \quad 0 < x < L, \ t > 0
\]

in which \(T = T(x, t)\) is the temperature at position \(x\) at time \(t\), \(g = g(x, t)\) is the heat source distribution which will be used to control the temperature throughout the media, \(k = k(x)\) is the thermal conductivity, \(\rho = \rho(x)\) is the mass density and \(C_v\) is the specific heat of the medium. Equation (1) is a linear two-point boundary value problem, which is subject to the generalized boundary conditions

\[
k_1 \frac{\partial T}{\partial x} + h_1 (T - T_0) = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad k_2 \frac{\partial T}{\partial x} + h_2 (T - T_L) = 0 \quad \text{at} \quad x = L.
\]

and the uniform temperature initial condition \(T(x,0) = T_0\).

Let us now expand the temperature in a series, and write

\[T(x,t) = \sum_{n=1}^{\infty} T_n(x) \phi_n(t)\]

where \(X_i\) is the normalization integral solution of the transcendental eigenfunctions satisfying

\[
\beta_i^2 - (H_i H_j) \tan \beta_i = 0
\]

subject to the homogeneous boundary conditions.

The eigenvalues \(\beta_i\) are determined by the orthogonality

\[
\int_0^L \rho C_v \phi_n(x) \phi_m(x) dx = \delta_{nm}
\]

where \(\delta_{ij}\) is the Kronecker delta.

Similar to Eq (1),

\[
g(x,t) = \sum_{n=1}^{\infty} g_n \phi_n(t)
\]

in which \(g_0\) is the initial condition.

We now take on both sides of the system the transform of the time derivatives.

\[
\mathcal{F}\{\frac{\partial \phi_n}{\partial t}\} = s \mathcal{F}\{\phi_n\}
\]

the transformed boundary conditions are obtained by applying the transform to the boundary conditions (2).

in which the constraint is given by

\[
\mathcal{F}\{k \frac{\partial \phi_n}{\partial x}\} = s k \mathcal{F}\{\phi_n\}
\]

The expansions of the solutions of the heat conduction problem, in terms of the eigenfunctions, \(\phi_n\) are given by

\[
\phi_n = \mathcal{F}^{-1}\left\{\mathcal{F}\{\phi_n\}\right\} = \mathcal{F}^{-1}\left\{\mathcal{F}\left\{T_0 \phi_n(t)\right\}\right\}
\]

in which the initial condition is transformed.

1 The term control spillover is defined in Eq. (19) later in the paper.

2 In the one-dimensional heat conduction problem, the discrete control inputs are plane surface heat sources.
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\[
\tau(x,t) = \sum_{i} \frac{X_i(x)}{M_i} \tau_i(t) \quad \text{where} \quad \tau_i(t) = \int_{x}^{\infty} X_i(x') \tau(x',t) \, dx', \quad \text{for } i = 1, 2, ..., \]

where \( X_i \) is the \( i \)th system eigenfunction, \( \tau_i \) is the \( i \)th transform of the temperature, and \( M_i \) is the \( i \)th normalization integral. In fact Eq.’s (3a) and (3b) define the integral transform pair associated with the solution of the transient heat conduction problem defined above (4). Assuming constant \( k \), \( \rho \), and \( C_p \) the eigenfunctions satisfy the eigenvalue problem

\[
\frac{d^2 X_i}{dx^2} + \beta_i^2 X_i = 0, \quad \text{for } 0 < x < L \quad \quad i = 1, 2, ...
\]

subject to the homogenous boundary conditions

\[
k_1 \frac{dX_i}{dx} - h_1 X_i = 0 \quad \text{at } x = 0 \quad \text{and} \quad k_2 \frac{dX_i}{dx} + h_2 X_i = 0 \quad \text{at } x = L
\]

(4b)

The eigenvalues \( \beta_i \) are the positive roots of the transcendental equation

\[ \beta^2 = H_1 \tan \beta L = 2 \beta (H_1 + H_2) \]

where \( H_1 = h_1/k \) and \( H_2 = h_2/k \). In addition, the eigenfunctions satisfy the orthogonality condition

\[
\int_{0}^{L} X_i(x) X_j(x) \, dx = M_i \delta_{ij} \quad i = 1, 2, ... \quad j = 1, 2, ...
\]

(4c)

where \( \delta_{ij} \) is the Kronecker delta.

Similar to Eq. (3), we expand the heat source distribution in a series, and write

\[
g(x,t) = \sum_{i} \frac{X_i(x)}{M_i} \tilde{g}_i(t) \quad \text{where} \quad \tilde{\tau}_i(t) = \int_{0}^{L} X_i(x') g(x',t) \, dx', \quad \text{for } i = 1, 2, ...
\]

(5a,b)

in which \( \tilde{\tau}_i \) is the \( i \)th transform of the heat source distribution.

We now take the integral transform of the problem defined by Eq.’s (1) and (2); namely we operate on both sides of the equation with the operator \( \int_{0}^{L} \, dx \), perform integration by parts and utilize the boundary conditions (2) to obtain the following infinite set of first order initial value problems for the transform of the temperature \( \tilde{\tau}_i \):

\[
\frac{d\tilde{\tau}_i}{dt} + \alpha \beta_i^2 \tilde{\tau}_i = \frac{\alpha}{k} (\tilde{\tau}_i + B_i), \quad i = 1, 2, ...
\]

(6a)

in which the contribution of the nonhomogeneous terms of the boundary conditions is denoted by \( B_i \) and given by

\[
B_i = X_i(0) h_1 T_{in} + X_i(L) h_2 T_{in}
\]

(6b)

The expansions of the temperature and of the heat source distribution are useful as an engineering tool as we find from Eq.’s (6). Indeed, the expansions of the temperature and the heat source distribution lead to a one-to-one correspondence between the components of temperature and the components of heat source distribution. This correspondence forms the basis for modal design. The coefficients \( \tilde{\tau}_i \) and \( \tilde{g}_i \) express the degrees to which the mode \( X_i \) participates in the overall distributions \( \tau \) and \( g \), respectively, and since Eq.’s (6) are uncoupled, the \( i \)th mode of heat can effect only the \( i \)th mode of temperature. That is, \( \tilde{\tau}_i \) has
no effect on \( T \), when \( i \) and \( j \) are different. Equations (6) will now be called the \textit{modal equations of heat conduction}.

\textbf{The Application of Modal Control to Heat Conduction}

The engineer, when solving the control problem of heat conduction can work directly, either with the governing equations of heat conduction (1), or with the modal equations of heat conduction (6). Modal control refers to a solution of the control problem using the modal equations of heat conduction.\(^3\) Using modal control our first step is to truncate the infinite set of equations (6) to \( N \) modes in which \( N \) indicates the number of modes of temperature significantly participating in the system response. The remaining modes are assumed to participate only insignificantly in the system response. They are referred to as the residual modes. The number \( N \) depends on the initial conditions and on the nature of the heat sources, which have yet to be designed. Never the less, \( N \) must be selected first, and if it becomes apparent that the level of control spillover into the residual modes is prohibitive, then \( N \) can be increased. Likewise, if it turns out that fewer modes than \( N \) participate in the response, then \( N \) can be decreased, and the engineering design process is repeated.

Next, working with Eq.'s (6), we develop an algorithm that governs the transforms of the heat source distribution. We express the \( i \text{th} \) transform of the heat source distribution \( \mathcal{Q}_i(t) \) as a function of the \( i \text{th} \) transform of the temperature \( \bar{T}_i(t) \). Note that it is natural to exclude the functional dependence between \( \mathcal{Q}_i(t) \) and \( \bar{T}_i(t) \), for different \( i \) and \( j \), since \( \bar{T}_i(t) \) is not effected by \( \mathcal{Q}_j(t) \) as observed in Eq. (6). In functional form, we write

\[ \mathcal{Q}_i = f_i(\bar{T}_i), \quad i = 1, 2, ..., N. \]  

(7)

An algorithm in the form of Eq. (7) is known as \textit{Independent Modal-Space Control} \(^1\). In general, the algorithm is a function depending only on our desire for certain steady state and transient properties of the temperature distribution. In this paper, we shall restrict our attention to linear functions of the form

\[ \mathcal{Q}_i = -\bar{\alpha}_i(\bar{T}_i - \bar{T}_\infty), \quad i = 1, 2, ..., N \]  

where \( \bar{\alpha}_i \) is a design constant to be determined, and \( \bar{T}_\infty \) refers to the transform of the desired temperature distribution. We will now examine separately, the implementation of Eq. (8) using distributed heat sources and discrete heat sources.

\textbf{Distributed Control}

Distributed control refers to heat source distributions that are continuous in space functions. Although, distributed control is generally difficult to implement, the solution of the heat transfer control problem employing distributed control lends insight into the solution of the control problem employing discrete heat sources. In the case of distributed control, the number of controlled modes \( N \) includes the

\[ ^3\text{In contrast, the method of direct feedback uses Eq. (1). The relationship between these two methods will be brought out later in the paper.} \]
full set of modes, that is \( N \) is infinite. Substituting Eq. (8) into Eq. (6), and solving for the modal equations of motion, we obtain

\[
\ddot{T}_i(t) = \left[ \ddot{T}_i(0) - \frac{\alpha T_m + \beta_i}{k \beta_i^2 + A} \right] \exp \left( -\frac{\alpha + \frac{A}{k}}{k} \right) + \frac{\alpha T_m + \beta_i}{k \beta_i^2 + A}.
\]

In Eq. (9) we now let \( \tilde{\alpha} \gg k \beta_i^2 \) which indicates that the applied heat source distribution damps out the transforms of the temperature at rates, \( \alpha \tilde{A}/k \), that are large compared to their natural damping rates, \( \alpha \beta_i^2 \). The condition \( \tilde{\alpha} \gg k \beta_i^2 \) is actually impossible to satisfy as \( i \) becomes large, since \( \beta_i^2 \) increases with increasing \( i \). Nevertheless, we shall obtain an approximate expression of the temperature distribution which will be useful for the interpretation of the design constants \( \tilde{A} \) and \( \tilde{T}_m \) in equation (8). We will also assume that \( \tilde{\alpha} = B \), that is, the energy input is much greater than any effect from the nonhomogeneous boundary terms. We also let the damping rates be equal to each other, that is, let \( \tilde{\alpha} = \tilde{\alpha} = \ldots = \tilde{A} \). Applying these assumptions and substituting Eq. (9) into Eq. (3), we obtain in closed form the uniformly damped solution of the heat conduction control problem employing distributed control

\[
T(x,t) = T(x,0) \exp \left( -\frac{\alpha}{k} \tilde{A} t \right) + T_o(x) \left[ 1 - \exp \left( -\frac{\alpha}{k} \tilde{A} t \right) \right]
\]

in which \( T(x,0) \) is the initial temperature distribution. The desired temperature distribution, \( T_o(x) \), is given by

\[
T_o(x) = \sum_{i=1}^{N} \frac{X_i(x)}{N_i} T_{\rho i} \text{ where } T_{\rho i} = \int_0^1 X_i(x') T_{\rho}(x') \, dx', \text{ for } i = 1, 2, ..., (11a,b)
\]

We now recognize from Eq. (10) that the design parameter \( \tilde{A} \) can be selected by the engineer on the basis of the desired damping rate, \( \alpha \tilde{A}/k \), which in turn governs the rate at which the temperature distribution converges to the desired temperature distribution \( T_o(x) \), that is, the steady state solution to Eq. (10).

**Discrete Control**

In practice, we may be confined to using discrete heat sources, which are expressed in the form

\[
g = \sum_{i=1}^{N} G_i \delta_i
\]

in which \( G_i = G_i(t) \) denotes the strength of a plane surface heat source at \( x = x_i \); and \( \delta = \delta(x-x_i) \) is the spatial Dirac delta function at \( x_i \). Notice in Eq. (12) that the summation is taken to \( N \), indicating that \( N \) discrete heat sources will be employed to control \( N \) modes of temperature. Taking the number of discrete heat sources not less than the number of controlled modes is an accepted engineering practice that is justified when we look at the alternatives. Using substantially fewer heat sources than modes participating in the response leads to solutions of the heat conduction control problem that exhibit such unattractive properties as undesirable transient and steady state temperatures, undesirable temperature distributions and solutions that are sensitive to the system parameters. Substituting Eq.'s (8) and (12) into the first \( N \) of equations (5b) we obtain
\[ \sum_{j=1}^{N} X_j G_i = -\overline{\lambda}_i (\overline{T}_j - \overline{T}_{\beta j}), \text{ for } i = 1, 2, \ldots, N \]  
(13)

in which \( X_j = X_j(x_j) \). Solving for \( G_i \) in Eq. (13) yields the algorithm that governs the strength of the discrete heat sources

\[ G_i = -\sum_{j=1}^{N} Y_{ij} \overline{\lambda}_i (\overline{T}_j - \overline{T}_{\beta j}), \text{ for } i = 1, 2, \ldots, N \]  
(14)

in which \( Y_{ij} \) are the elements of the inverse of the \( N\times N \) matrix having elements \( X_{ij} \).

The algorithm given by Eq. (14) requires knowledge of the transforms of the temperature \( \overline{T}_j \) and the desired temperature \( \overline{T}_{\beta j} \). For simplicity, let us measure the temperature at the locations of the discrete heat sources, that is we collocate the discrete heat sources and the temperature measurements. Truncating the series given in Eq. (3a) and (11a) to \( N \) terms, we get

\[ T_i = \sum_{j=1}^{N} \frac{X_j}{M_j} \overline{T}_j \quad \text{and} \quad \overline{T}_{\beta i} = \sum_{j=1}^{N} \frac{X_j}{M_j} \overline{T}_{\beta j} \]  
(15a,b)

where \( T_i = T(x_i, t) \) and \( \overline{T}_{\beta i} = T_{\beta}(x_i) \), which yields upon matrix inversion

\[ \overline{T}_i = M_i \sum_{j=1}^{N} Y_{ij} \overline{T}_j \quad \text{and} \quad \overline{T}_{\beta i} = M_i \sum_{j=1}^{N} Y_{ij} \overline{T}_{\beta j} \text{ for } i = 1, 2, \ldots, N. \]  
(16a,b)

Substituting Eq. (16) into (14), we obtain an algorithm that controls the damping rate (i.e. the rate at which the medium approaches the desired temperature) of each individual mode of temperature independently,

\[ G_i = -\sum_{j=1}^{N} A_{ij} (T_j - T_{\beta j}) \]  
for \( i = 1, 2, \ldots, N \).  
(17)

in which the design parameters \( A_{ij} \) are defined by

\[ A_{ij} = \sum_{j=1}^{N} M_j \overline{\lambda}_j Y_{ij} Y_{j} \]  
for \( i = 1, 2, \ldots, N \).  
(18)

Finally, for completeness, let us show that the residual modes are excited by the discrete heat sources. Substituting Eq. (12) into (5b) for \( i = N+1, N+2, \ldots, \) we obtain the transforms of the heat sources associated with the residual modes

\[ \overline{F}_i = \sum_{j=1}^{N} X_j G_j \]  
for \( i = N+1, N+2, \ldots, \).  
(19)

in which \( G_j \) were determined earlier by Eq. (17) and (18). The excitation of the residual modes by \( G_j \) is called control spillover.

**Direct Feedback Control**

Direct feedback control refers to a solution of the heat conduction control problem that employs heat sources governed by algorithms that are independent of each other. In Eq. (17), this implies that the

...
design matrix $A_j$ is diagonal, that is $A_j = A_j \delta_j$. Among the alternatives discussed so far, namely distributed (modal) control, discrete (modal) control and direct feedback control, direct feedback control is the simplest to implement. Of immediate interest is to develop direct feedback control and its relationship to modal control. It should be noted that direct feedback control could stand on its own and would not need to be related to modal control. However, the relationship between direct feedback control and modal control will lead to conditions that will illuminate the types of performances that one can achieve using direct feedback control.

We start by reconsidering the linear feedback control algorithm, Eq. (17)

$$ G_i = - \sum_{i=1}^{N} A_i (T_j - T_{0i}) \quad \text{for } i = 1, 2, \ldots, N $$

(20)

Substituting Eq.'s (15a and b) into Eq. (20), and the resulting expression into Eq. (12) and then Eq. (5b), we obtain the linear modal control algorithm

$$ g_i = - \sum_{i=1}^{N} A_i (\bar{T}_j - \bar{T}_{0i}) \quad \text{for } i = 1, 2, \ldots, N $$

(21)

in which

$$ \bar{A}_i = \sum_{j=1}^{N} X_j A_j M_j^T X_j \quad \text{for } i = 1, 2, \ldots, N. $$

(22)

Equation (20) reduces to direct feedback control when $A_j = A_0 \delta_j$ and Eq. (21) reduces to independent modal space control, Eq. (8), when we let $\bar{A}_i = \bar{A} \delta_j$.

We next seek the simultaneous conditions under which independent modal space control is direct feedback control. Letting $A_j = A_0 \delta_j$ and $\bar{A}_i = \bar{A} \delta_j$ in Eq. (22), we obtain

$$ \bar{A} \delta_j = \sum_{j=1}^{N} X_j A_j M_j^T X_j \quad \text{for } i = 1, 2, \ldots, N. $$

(23)

Equation (23) represents $N^2$ equations and $N$ unknowns, in which case an exact solution would usually be unachievable. However, it turns out that Eq. (23) can be satisfied exactly when the discrete heat sources are located appropriately, and when the heat sources are damped appropriately. Specifically, we let the heat sources be located at the zeros of the $N + 1^{st}$ mode, that is the locations of the heat sources $x_i (i = 1, 2, \ldots, N)$ solve the equation

$$ X_{n, i} (x_i) = 0 \quad \text{for } i = 1, 2, \ldots, N. $$

(24)

Furthermore, we prescribe $\bar{A} = \bar{A}_0 = \ldots = \bar{A}_n = \bar{A}$, that is, uniform heat source damping [5,6]. Letting $i = j$ in Eq. (23), and inverting the result we get

$$ A_i = \sum_{j=1}^{N} (Z_j M_j) \bar{A} \quad \text{for } i = 1, 2, \ldots, N. $$

(25)
in which $Z_a$ are the elements of the inverse of the $N \times N$ matrix having entries $X_i^j$. Furthermore, letting $i \neq j$ in Eq. (23), we get

$$0 = \sum_{i=1}^{N} X_i A_i M_i X_a \text{ for } i, j = 1, 2, ..., N \tag{26}$$

which is satisfied identically. Equation (26) can be regarded as a property of the zeros of the eigenfunctions. The satisfaction of Eq. (26) is restricted to constant property coefficients in Eq.'s (1) and (2). For more details refer to reference [6].

The above demonstrates that direct feedback control is natural when the heat sources are designed to uniformly dampen the temperature, and when the heat sources are located at the zeros of the $N+1^\text{st}$ mode. The damping will be non-uniform and the modes will be coupled (i.e. $A_{ij} \neq 0$ for $i \neq j$) when the locations of the heat sources, $x_i$ ($i = 1, 2, ..., N$) and the design constants $A_i$ ($i = 1, 2, ..., N$) deviate from the values otherwise obtained from Eq.'s (24) and (25).

We now illustrate modal control, and in particular direct feedback control with the following two examples.

**Example**

Consider the one-dimensional heat conduction control problem

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{k} \sum_{i=1}^{N} G_i \delta_i = \frac{1}{\alpha} \frac{\partial T}{\partial t}, \quad 0 < x < 1, t > 0 \tag{27}$$

where $G_i$ is defined in Eq. (17), in which $A_j$ is defined in Eq. (18). The associated boundary conditions are

$$\frac{\partial T}{\partial x} = 0 \text{ at } x = 0 \quad \text{and} \quad -\frac{1}{k} \frac{\partial T}{\partial x} = h(T - T_\infty) \text{ at } x = 1, t > 0 \tag{28}$$

and the initial condition is

$$T(x, 0) = 0 \text{ for } t = 0. \tag{29}$$

The desired temperature distribution in this problem is the constant $T_0 = 5^\circ C$. Equation (27) is rewritten as

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{k} \sum_{i=1}^{N} A_i (T_i - T_{\infty}) \delta(x - x_i) = \frac{1}{\alpha} \frac{\partial T}{\partial t}, \quad 0 < x < 1, t > 0 \tag{30}$$

At this point the designer must decide whether uniform damping is needed or appropriate. For these examples we choose uniform damping and, hence, apply the algorithms developed for direct feedback. We let $A_j = A_i \delta_i$, where $A_i$ is defined in Eq. (25).

We now consider the three cases in which we apply $N = 2, 5$, and 10 discrete heat sources. The physical properties are $k = 100 \text{ W/mK}$, $\alpha = 3.412 \times 10^{-4} \text{ m}^2/\text{s}$, $h = 20 \text{ W/m}^2\text{K}$ and $T_\infty = 0^\circ C$. The system eigenvectors and eigenvalues are given by [4] $X_i = \cos(\beta_i x)$ where $\beta_i$ are solutions of $\beta_i \tan(\beta_i) = h/k$. The normalization constant is $M_i = 1/2(\beta_i^2 + (h/k)^2 + h/k)(\beta_i^2 + (h/k)^2)^{-1}$. The first four eigenvalues are

$$\beta_1 = 0.43 \text{ eigenfunc} \quad \text{We n} \quad \text{desired te} \quad \text{required :} \quad \begin{array}{c}
\lambda_1 = 2.25 \\
A_1 = 112. \end{array} \tag{25}$$

Figur: of Eq.'s : \quad \text{condition : heat sour at the rig location distributed in other temp : with an solution are dom excessiv this effe.}

In this p conduct: Accordi: engine: behavio: control: and the : prescrib
Furthermore, letting the zeros of the terms in Eq.'s (1) and (2) be designed to be zero of the \( N+1 \) term for \( i \neq j \) when the \( N \) deviate from the following two

\[ (27) \]
\[ \text{boundary conditions} \]

\[ (28) \]
\[ (29) \]
\[ \text{in (27) is rewritten} \]

appropriate. For these direct feedback.

heat sources. The 0°C. The system of \( \beta, \tan \beta = h/k \). 

Accordingly, the zeros of the \( N + 1 \)th eigenfunction, say for two heat sources (\( N = 2 \)), are \( x_1 = 0.2487 \) and \( x_2 = 0.7462 \).

We now specify \( \bar{A} \) in Eq. (25), depending on the desired rate at which the medium approaches the desired temperature. One convenient way of choosing \( \bar{A} \) is by first selecting a settling time, \( t_e \); the time required for the medium to approach 90% of the desired temperature. A good approximation to evaluate \( \bar{A} \) is to set the last term, \( 1 - \exp \left( -\alpha \bar{A} t \right) \), in Eq. (10) equal to 0.9 and \( t = t_e \). Now we chose \( \bar{A} = 2.25 \times 10^4 \) J/m²K, corresponding to a 30 second settling time; it yields \( A_1 = 111,377 \) and \( A_2 = 112,746 \) W/m²K for two heat sources. The \( A_i \)'s for any number of heat sources are calculated from Eq. (25).

Figure 1(a-c) shows the temperature distribution as a function of time, determined from the solution of Eq.'s (27)-(29) for \( N = 2, 5 \) and 10 heat sources respectively. The temperature increases from the initial condition to the desired temperature in an uniform manner, and within 30 seconds the temperatures at the heat sources have reached 90% of the desired values. The final steady state solution has a small deviation at the right boundary. This is expected since the convective cooling drives the temperature down at this location unless a heat source is located on the boundary itself. At any specified time, the temperature distribution with 10 controllers is much closer to the desired temperature than that with 5 or 2 controllers. In other words, the application of additional heat sources tends to decrease the magnitude of the temperature gradients. Figure 1(d) shows that the energy input to reach steady state decreases slightly with an increasing number of heat sources up to about 5 to 6 heat sources. Upon closer examination of the solution of the transient heat conduction problem, Eq.'s (27) – (29), we find that the first 6 eigenfunctions are dominant. Consequently we select 6 heat sources. Choosing fewer heat sources would cause an excessive level of control spillover into the residual modes and more energy input over time to overcome this effect.

**Summary**

In this paper, a methodology was developed for controlling heat transfer in one dimensional transient heat conduction problems by utilizing basic concepts used for controlling the vibration of mechanical systems. Accordingly, this paper introduced the important concept of modes of temperature and their role in the engineering design of controlled heat sources, which in turn influenced both the spatial and temporal behavior of the solution. The concepts were developed in a progression beginning with distributed control, and culminating with direct feedback control. Finally, a specific example determined the power and the location of the plane surface heat sources in a plate designed to maintain the plate temperature at a prescribed level.
FIG. 1
Temperature distribution as a function of time for a) N = 2, b) N = 5,
c) N = 10; d) Total energy input as a function of number of heat sources.

References


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