One-Dimensional Large Deformation Theory for Charge-Carrying Bodies

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Abstract

A numerical method is developed to predict the charge distribution on conductive surfaces resulting in elastic deformations. As an initial study, this research effort focuses on one-dimensional conductors. Minimization of the potential energy associated with a charge distribution yields the static deformation shapes. Charge distributions are computed for one-, two- and three-dimensional surfaces. Deformations are computed for one-dimensional members. Results illustrate the feasibility of the method which can be extended to higher-dimensional and dynamic cases.

1. Introduction

The field of electrostatics dates back to the classical works of Coulomb and since then has been honored by many historic works. Likewise, the field of mechanics dates back just as far, and within its domain contains a great many classical works. In contrast, the coupling of these two fields is of recent vintage. Indeed, coupled problems of electrostatics and mechanics are found in contemporary engineering applications. Examples of these include electrostatic speakers [1,2], scientific instruments [3,4], and space-based antennas [5,6]. Upon reviewing these works, the authors have found that the mathematical foundation of the coupled electrostatics-mechanics problem is severely limited. In fact, it can be speculated that these applications and others are currently limited by the inavailability of a strong mathematical foundation relating to this coupled problem. This paper is an attempt at strengthening such a foundation.

Section 2 formulates the governing equations. Section 3 describes the discretization process necessary in obtaining the numerical solutions. Section 4 describes the associated potential energy minimization problem. Sections 5 and 6 provide illustrative solutions and Section 7 gives some final remarks.

2. Governing Equations

The electric potential at point \( p \) is given by

\[
V(p) = k \int_{Q} \frac{dq(p')}{|r(p) - r(p')|} \tag{1}
\]

where \( dq(p') \) is the charge at point \( p' \), \( r(p) \) is the position vector of point \( p \) and \( k = \frac{1}{4\pi \varepsilon_0} \approx 9 \times 10^9 \) volt meter/coulomb [7]. The integration is carried out over all of the charges present. Next, consider a one-dimensional conductive member. The potential energy \( U \) of the member is [8,9]

\[
U = U_E + U_G + U_S \tag{2}
\]

where \( U_E \) denotes the electrostatic component, \( U_G \) the gravitational component and \( U_S \) denotes the strain component. The electrostatic energy and the gravitational energy are given by

\[
U_E = \frac{1}{2} \int_{Q} V(p) dq(p) , \quad U_G = -\int_{M} r(p) \cdot dm(p) \cdot g \tag{3a,b}
\]

where \( g \) is the gravity vector and \( dm(p) \) is the mass at point \( p \). The mechanical strain energy is given by

\[
U_S = \frac{1}{2} \int_{L} EI \left( \frac{1}{R(s)} - \frac{1}{R_0(s)} \right)^2 ds \tag{3c}
\]

where \( R_0(s) \) and \( R(s) \) are the radii of curvature at point \( s \) of the undeformed and deformed members, respectively. The integration is carried out along the neutral axis of the member, and \( L \) is its arc length, which is assumed not to change under the deformation (see Fig. 1).

![Figure 1: Discretized one-dimensional member](image-url)
3 Spatial Discretization of the Governing Equations

To approximate the charge distribution on a conducting surface, we discretize the surface into flat rectangular regions over which the charge density is constant [10]. From Eq. (1), we obtain

\[
V(p_i) = k \sum_{j=1}^{n} \frac{q(p_j)}{r(p_i) - r(p_j)} + k \sum_{j=1}^{n} \frac{dq(p)}{r(p_j) - r(p_j)}
\]

where

\[
P_{ij} = \frac{k}{r(p_i) - r(p_j)} , \quad i \neq j
\]

in which \( r(p_i) - r(p_j) \) = \( \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} \).

For \( i = j \), we carry out the integration in Eq. (4) and obtain

\[
P_{ii} = k \ln \left( \frac{a + \sqrt{a^2 + b^2}}{a} \right) + k \ln \left( \frac{b + \sqrt{a^2 + b^2}}{b} \right)
\]

where \( a \) and \( b \) are the lengths of the sides of the \( i \)-th element (Ref. [8]). When \( a = b \), \( P_{ii} = 3.5255 \frac{k}{a} \).

Equation (4) is written more compactly in the vector-matrix form

\[
V = P Q
\]

where \( V = (V_1, V_2, ..., V_n)^T \), \( Q = (q_1, q_2, ..., q_n)^T \), \( V_i = V(p_i) \), and \( q_i = q(p_i) \). Inverting Eq. (6a), we obtain the charges

\[
Q = C V
\]

where \( C = P^{-1} \). Since every point on a conductor has the same potential, each rectangular region in the discretized surface is assumed to have the same potential. Turning to the electrostatic energy in the system, we treat the charge distribution similarly, and obtain

\[
U_E = \frac{1}{2} \sum_{i=1}^{n} V(p_i)q(p_i) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} P_{ij}q(p_i)q(p_j)
\]

Letting \( d_{ij} = \| r(p_i) - r(p_j) \| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \)

and \( q_i = q(p_i) \) where

\[
x_i = \sum_{s=1}^{i} h_s \cos \theta_s , \quad y_i = \sum_{s=1}^{i} h_s \sin \theta_s
\]

we arrive at

\[
U_E = k \sum_{i=1}^{n} \sum_{j=i+1}^{n} q_i q_j d_{ij}
\]

The gravitational energy is discretized by substituting \( g = y_i g \) and \( dm(p_i) = m_i \) into Eq. (3b) to yield

\[
U_G = g \sum_{i=1}^{n} m_i y_i
\]

Finally turning our attention to the strain energy, since \( R(s)d\theta = ds \), and letting

\[
\frac{d\theta}{ds} = \frac{\theta_i - \theta_{i-1}}{s_i - s_{i-1}} = \frac{\theta_i - \theta_{i-1}}{h_i} , \quad ds = \frac{\phi_i - \phi_{i-1}}{h_i}
\]

where \( h_i = s_i - s_{i-1} \), we obtain (see Fig. 1)

\[
U_S = \frac{1}{2} \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} \frac{\phi_i - \phi_{i-1}}{h_i} \left[ \left( \theta_i - \theta_{i-1} - (\phi_i - \phi_{i-1}) \right)^2 \right] ds
\]

where we assign \( \theta_0 = \phi_0 = 0 \). Equation (6b) gives the charge distribution for a system of \( n \) points, each at a specified electric potential (which may vary from point to point if we wish to model a non-conducting system). Equation (9) gives the electrostatic energy contained in the system. To determine the gravitational energy in the system we may use equation (10), and the mechanical strain energy for our one-dimensional member is given by equation (12).

4 Potential Energy Minimization

We seek to minimize the potential energy of the system. Toward this end, we employ Newton's Method [11]. At the \( r \)-th iteration
\[ \theta_i(r + 1) = \theta_i(r) - \sum_{j=1}^{n} A_{ij} F_j \]  \hspace{1cm} (13)

where \( A = K^{-1} \) and where

\[ F_j = \frac{\partial U}{\partial \theta_j} = \frac{\partial U_E}{\partial \theta_j} + \frac{\partial U_G}{\partial \theta_j} + \frac{\partial U_S}{\partial \theta_j} \]  \hspace{1cm} (14a)

\[ K_{ij} = k_{ij} = \frac{\partial^2 U}{\partial \theta_i \partial \theta_j} = \frac{\partial^2 U_E}{\partial \theta_i \partial \theta_j} + \frac{\partial^2 U_G}{\partial \theta_i \partial \theta_j} + \frac{\partial^2 U_S}{\partial \theta_i \partial \theta_j} \]  \hspace{1cm} (14b)

The charge distribution is fixed during this iteration. The next step is to compute a new charge distribution using Eq. (6b). It can be shown that this step is identical to minimizing the potential energy while holding the displacements fixed. Steps 1 and 2 are repeated until we converge to the equilibrium charge and equilibrium displacements. Energy monotonically decreases with the iteration number so convergence is guaranteed. The gradients of equations (14) are given in Ref. [15].

5 Undeformed shapes

In this Section the method for computing charge distributions on surfaces is applied to three-dimensional conducting bodies of prescribed shapes. The applied voltage is also prescribed. In particular we apply the discretization method described in Section 3 to a round disc and a cone. A line segment and square plate are considered in Ref. [15].

5.1 Round Disc

Let’s look at a round disc of radius \( R = 1 \). In this case we face the problem of approximating a round object with rectangular elements. Also, there exists an analytical solution to which we can compare our solution [12].

For the discretization we first divide the disc into \( n \) concentric rings, each of width \( h \). The \( i \)-th ring is then divided into \( m_i \) elements, each having arc length about their center \( \ell_i \) (see Fig. 2). The number of elements in the \( i \)-th ring is selected so that \( \ell_i \) is as close to \( h \) as possible. The radius at the center of the ring is \( r_i = (i - 1) h + h/2 \) and its circumference is \( C = 2 \pi r_i = m_i \ell_i \). We let \( m_i = \text{nint} \left( 2 \pi [(i - 1) + \frac{1}{2}] \right) \) (where \( \text{nint}(x) \) is the nearest integer to \( x \)) so that \( \ell_i = 2 \pi [(i - 1) + \frac{1}{2}] \frac{h}{m_i} \). Next, we let \( \Delta \theta = \frac{2 \pi}{m_i} \), so that \( \theta_{ij} = j \Delta \theta ; \ j = 1, 2, \ldots, m_i \). Then \( x_{ij} = r_i \cos \theta_{ij} ; \ y_{ij} = r_i \sin \theta_{ij} \), where the double index \( ij \) has been introduced for simplicity. The results are shown in Fig. 3. The solid line represents the analytical solution due to Rao, et al. [12]. Numerical solutions are plotted for \( n = 3, 5, 10, 20 \) which correspond to the total number of elements \( N = 28, 78, 314, \text{ and } 1257 \).

Figure 2: Discretization of round disc

\[ \text{Figure 3: Charge distribution on round disc (V = 10^7) for n = 3, 5, 10, 20 with comparison to analytical (q_iN/\pi R \text{V e}_0 \text{ vs. } r/R)} \]

5.2 Cone

As a 3-dimensional example we look at a cone. The discretization here is similar to that of the disc but with some added complexity. We choose a cone of height \( H = 2 \) and bottom radius \( R = 1 \) (see Fig. 5). Projecting the right half of the cone onto the vertical plane we have \( \gamma = \tan^{-1} (R/H) \). We divide the vertical axis into strips each of thickness \( \Delta z = \frac{H}{n} \) such that \( \bar{z}_j = \frac{H}{n} (i - 1) \). Corresponding to this, along the cone each strip has width

\[ \text{Figure 4: Discretization of cone} \]
h = Δz/cos γ. The radius of the cone at height z_i is r_i = R\left(1 - \frac{z_i}{h}\right), so that m_i = n(2\pi \frac{r_i}{h}). As with the disc, this strip is divided into m_i \times n_d (i) elements of length \ell_i, so that m_i \ell_i = 2\pi r_i. Letting m_i = \text{nint}\left(\frac{2\pi r_i}{h}\right), we obtain m_i elements which can be considered nearly rectangular. To obtain the positions for these elements we set \Delta \theta_i = \frac{2\pi}{m_i} and \theta_{i+j} = j\Delta \theta_i; \quad j = 1, 2, ..., m_i. Then x_{ij} = r_i \cos \theta_{ij}, y_{ij} = r_i \sin \theta_{ij}, and z_{ij} = \frac{1}{n} (i - 1). Figure 5 shows the charge distribution along the side of the cone from top to bottom for n=33.

6 Deformed Shapes

6.1 Uncharged beam

We begin by determining the accuracy of our discretization method for a member which carries no charge. In this case Q = 0, φ = 0 in Eqs. (6b) and (12) and we consider a cantilever beam of length L, mass M, and stiffness EI. Figure 6 compares our result for n = 80 with that of a finite-element solution due to Yang [13]. A charged beam is considered in Ref. [15].

![Figure 6: Charge distribution on cone](image)

VI Deformed Shapes

6.2 Spiral

Next we consider the case of a spiral, which is described by the equation \eta = b\theta/\theta_{\text{max}}. In this case no gravity is present. We wish to approximate the spiral as made up of flat, square elements, all of the same size. We begin by dividing the spiral into n−1 pieces, each of arc length \ell = L/(n−1). The general arc length is given by

\[ L = \int_{\theta_0}^{\theta_{\text{max}}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \]

where L is the arc length of the curve r(\theta) from \theta_0 to \theta_{\text{max}} (Ref. [14]). Note that \theta here is measured from the origin unlike the \theta in Fig. 1 (see Fig. 7). Letting \theta_0 = 0 we have

\[ \ell = b \int_{\theta_{\text{max}}}^{\theta_{\text{max}}(n-1)} \sqrt{\theta_{\text{max}}^2 + 1} + \ln \left(\theta_{\text{max}} + \sqrt{\theta_{\text{max}}^2 + 1}\right) \left(\theta_{\text{max}} - \frac{\theta_{\text{max}}^2 + 1}{\theta_{\text{max}} + \sqrt{\theta_{\text{max}}^2 + 1}}\right) (i - 1) \ell = 0 \]

To find \theta_i we must solve the nonlinear equation

\[ f(\theta_i) = b \left[\theta_i \sqrt{\theta_i^2 + 1} + \ln \left(\theta_i + \sqrt{\theta_i^2 + 1}\right)\right] - (i - 1) \ell = 0 \]

Letting \phi_i = r_i \cos \theta_i and \eta_i = \sin \theta_i, we arrive at

\[ \phi_i = \tan^{-1}\left(\frac{\eta_i - \eta_{i-1}}{\phi_i - \phi_{i-1}}\right) \]

Figure 8 shows how the spiral tends to unwind as it gets charged up. Square and triangular frames are considered in Ref. [15] along with an illustration of electrostatically induced snap-through behavior of a beam.

Figure 7: Comparison of n=80 solution (solid line) to Yang's solution (dashed line)

![Figure 7](image)

Figure 8: Discretization of spiral
7 Conclusions

This paper presented a method for computing charge distributions on three-dimensional conducting surfaces (with one- and two-dimensional surfaces as special cases) and for computing deformations of one-dimensional charge-carrying members.

The development began by presenting the relevant governing equations in integral form. In order to solve these equations for the various cases of interest, they were discretized. Next, an iterative numerical method was presented to be used in determining the deformations for any given one-dimensional member. Various examples of the methods were presented. The charge distribution on a round disc was computed and compared to the analytical solution. Close agreement between the two solutions was found. The deformation of an uncharged cantilever beam due to gravity was computed and compared to a finite-element solution. Again close agreement was found. Following this, deformations of various one-dimensional charge-carrying members were computed.

Areas of future research include extending the governing equations to include two- and three-dimensional members. This may require employing plate and shell theories in conjunction with finite-element discretization methods. Further, the research effort presented here represents static equilibrium analysis. Another area of future research, therefore, is to develop dynamic models for surface shape deformations due to charge distributions.

VIII. References


