PLANAR ELECTRODYNAMICS OF INTERCONNECTED CHARGED PARTICLES

by

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ABSTRACT

The planar electrodynamics of interconnected charged particles is investigated. The particle motion is conservative and governed by a set of nonlinear ordinary differential equations. Linearization about static equilibrium admits normal mode behavior. As tutorial examples, the dynamics of the electrostatic pendulum, the fixed charge electroscope and the electrostatic string are treated. Next, an experiment was conducted using charged spheres suspended by insulating strings to cancel the gravitational force. The measured equilibrium angles agree with the analytically predicted equilibrium angles, with a 0.16° error. The measured natural frequencies of oscillation agree with the predicted natural frequencies, with a 10-20% linearization error.

INTRODUCTION

The limitations in modifying a structure's gross performance (mass, size, damping, stiffness, slewing rates, etc.) by tailoring material properties (fiber reinforcements, braids, weaves, energy dissipating parts, etc.) has led to the demand for performance modifications by means of active components (pneumatic/hydraulic actuators, piezoelectric sensors/actuators, imbedded fiber-optic sensors, motors, etc.; see Refs. 1-6). Fundamentally, the forces that modify material

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properties and hence performance are internal electrical forces representing bonds between molecules. Within this context, design consists of judiciously arranging the molecules to yield the desired properties. Topologically, these arrangements consist of molecules forming material, composing a structure, sensors, actuators, computational elements, etc. The synthesis and reorganization of these elements into active (smart, adaptive, intelligent) structures is currently the focus of numerous engineering investigations (Refs. 7-10).

As previously mentioned the forces that modify performance are principally internal electrical bonds between molecules. In contrast, the presence of external electrical forces produced by free surface electrons, i.e., surface charge, is generally regarded as a liability and is suppressed by appropriate insulation and grounding techniques (Ref. 11). This paper suggests that surface charge, when appropriately controlled, can be turned from a liability into an asset. Potential benefits include reduction of thermally induced mechanical stresses caused by thermal gradients, reduced warping, elimination of moving mechanical components, varying equilibrium position capability (deployment, slewing, uncoiling, varying focal length, etc.). The technology is applicable to precise surfaces (lenses, mirrors, microwave/optical antennas, reflectors, acoustical membranes, etc.).

Within this context, this paper considers structures who's dynamic characteristics are predominantly derived from external free-electron forces. Toward that end, the structure can be represented either as a continuum, as discrete interconnected particles, or as a hybrid of the two. These structures will be referred to as electrodynamic structures and the study of such systems will be referred to as structural electrodynamics. Adopting a discrete particles approach, the next section formulates the equations governing the planar electrodynamics of interconnected charged particles. Then as tutorial examples, the paper describes the electrodynamics of the electrostatic pendulum, the fixed-charge electroscope and the electrostatic string. Next, we present an experiment comprised of interconnected charged spheres suspended by insulating strings to cancel the gravitational force. Measured equilibrium angles and natural frequencies are compared with the analytical predictions. Finally, a few concluding remarks are made.
DERIVATION OF THE EQUATIONS OF MOTION

We consider a planar electrodynamic system of n point charges suspended by insulated massless strings as shown in Fig. 1. As shown, inertially fixed charges, represented by black dots and suspended charges, represented by white dots, are enclosed in a grounded electrodynamic chamber. The position and velocity vectors of the point charges are expressed as functions of the independent coordinates

\[ \mathbf{r}_i = \mathbf{r}_i(\theta_1, \theta_2, \ldots, \theta_m), \quad \mathbf{\dot{r}}_i = \sum_{j=1}^{m} \frac{\partial \mathbf{r}_i}{\partial \theta_j} \dot{\theta}_j, \quad (i = 1, 2, \ldots, n) \]  \hspace{1cm} (1)

in which \( \mathbf{r}_i = x_i \mathbf{\hat{n}}_1 + y_i \mathbf{\hat{n}}_2 \) \((i = 1, 2, \ldots, n)\) denotes the position vector of the i-th point charge, \( \theta_j \) \((j = 1, 2, \ldots, m)\) denotes the j-th angular position relative to the \( \mathbf{\hat{n}}_1 \) axis about the \( \mathbf{\hat{n}}_3 \) axis, and overdots denote derivatives with respect to time. The kinetic energy of the system becomes

\[ T = \sum_{i=1}^{n} \frac{1}{2} m_i \mathbf{\dot{r}}_i \cdot \mathbf{\dot{r}}_i = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{m} m_i \frac{\partial \mathbf{r}_i}{\partial \theta_j} \cdot \frac{\partial \mathbf{r}_i}{\partial \theta_k} \dot{\theta}_j \dot{\theta}_k \]  \hspace{1cm} (2)

Figure 1. An example of an electrodynamic system
where \( m_i \) \((i = 1, 2, ..., n)\) denotes the mass of the \(i\)-th point charge. We assume that the plane of motion is perpendicular to the gravity vector as indicated by the shaded region in Fig. 1. The potential energy is composed of gravitational potential energy and electrical potential energy, given by

\[
U = U_e + U_g, \quad U_e = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{k q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad U_g = \sum_{i=1}^{n} \frac{m_i g}{2L} |\mathbf{r}_i - \mathbf{r}_j|^2
\]

in which \( q_i \) \((i = 1, 2, ..., n)\) denote fixed electrical charges, and \( k \) denotes the electrical constant of the medium. The magnitude of the gravitational acceleration is denoted by \( g \), the length of the massless suspending strings is denoted by \( L \) and the nominal positions of the free ends of the suspending strings are denoted by \( \mathbf{r}_i^1 = x_i^1 \hat{n}_1 + y_i^1 \hat{n}_2 \) \((i = 1, 2, ..., n)\). (The fixed end of the suspending strings are then located at \( \mathbf{r}_i^1 + L \hat{n}_3 \).) Invoking Lagrange's equations of motion for conservative systems (Ref. 12), we obtain the set of nonlinear ordinary differential equations of the form

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_k} \right) - \frac{\partial T}{\partial \theta_k} + \frac{\partial U}{\partial \theta_k} = 0, \quad (k = 1, 2, ..., m)
\]

Substituting Eqs. (2) and (3) into (4) and carrying out the necessary differentiations, yields the set of equations governing the motion of the planar electrodynamic system

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} m_j \frac{\partial \mathbf{r}_i}{\partial \theta_k} \cdot \left( \frac{\partial \mathbf{r}_i}{\partial \theta_j} + \frac{\partial^2 \mathbf{r}_i}{\partial \theta_j^2} \right) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{k q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|^2} \frac{\partial |\mathbf{r}_i - \mathbf{r}_j|}{\partial \theta_k}
\]

\[
+ \sum_{i=1}^{n} \frac{m_i g}{L} \frac{\partial}{\partial \theta_k} |\mathbf{r}_i - \mathbf{r}_i^1| \frac{\partial |\mathbf{r}_i - \mathbf{r}_i^1|}{\partial \theta_k} = 0 \quad (k = 1, 2, ..., m)
\]
Static Equilibrium Position

The static equilibrium position is found by letting \( \dot{\theta}_k(t) = \ddot{\theta}_k(t) = 0 \), \( (k = 1, 2, \ldots, m) \) in Eq. (5).

We obtain the nonlinear set of algebraic equations

\[
- \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{K_{ij} q_i q_j}{|r_i^0 - r_j^0|^2} \frac{\partial}{\partial \theta_k^0} |r_i^0 - r_j^0| + \sum_{i=1}^{n} \frac{m_i g}{L} \frac{\partial}{\partial \theta_k} |r_i^0 - r_i^1| \frac{\partial}{\partial \theta_k} |r_i^0 - r_i^1| = 0 ,
\]

\( (k = 1, 2, \ldots, m) \) \( (6) \)

in which \( r_i^0 \) \( (i = 1, 2, \ldots, n) \) and \( \theta_j^0 \) \( (j = 1, 2, \ldots, m) \) denote equilibrium positions.

The interest lies now in determining charges that yield a desirable equilibrium position.

Toward that end, the equilibrium position is prescribed and Eq. (6) represents a set of \( m \) quadratically nonlinear algebraic equations expressed in terms of unknown charges \( q_i \) \( (i = 1, 2, \ldots, n) \). Equation (6) is written as

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ijk} q_i q_j = d_k , \quad (k = 1, 2, \ldots, m) \]

\( (7) \)

in which

\[
a_{ijk} = \frac{k}{|r_i^0 - r_j^0|^2} \frac{\partial}{\partial \theta_k} |r_i^0 - r_j^0| , \quad d_k = \sum_{i=1}^{n} \frac{m_i g}{L} |r_i^0 - r_i^1| \frac{\partial}{\partial \theta_k} |r_i^0 - r_i^1| .
\]

The exact solution to Eq. (7) can be obtained numerically. As an alternative, an approximate solution can be obtained by a first-order perturbation analysis in which we let

\[
a_{ijk} = a_{ijk}^0 + a_{ijk}^1 , \quad q_i = q_i^0 + q_i^1 \]

\( (8) \)

where \( a_{ijk}^0 \) and \( q_i^0 \) are nominal quantities and \( a_{ijk}^1 \) and \( q_i^1 \) are perturbations. Substituting Eq. (8) into Eq. (7) and neglecting second-order terms, yields the set of linear algebraic equations

\[
\sum_{j=1}^{n} b_{ij} q_j^1 = c_i , \quad (i = 1, 2, \ldots, m) \]

\( (9) \)

where
\[ b_{ij} = 2 \sum_{k=1}^{n} a_{jki} q_k^0, \quad c_i = -\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jki} q_j^0 q_k^0 \]  

(10)

in which \( b_{ij} \) and \( c_i \) are known coefficients and \( q_j^1 \) are unknown perturbations in the electrical charges. The solution to Eq. (9) yields \( q_j^1 \), \((j = 1,2,\ldots,n)\). The perturbation analysis is not considered further in this paper. However, its usefulness further arises as a design tool and in the context of feedback control.

**Linearized electrodynamics**

The linearized electrodynamics relative to equilibrium is now considered by substituting the relative angular positions \( \eta_i(t) = \theta_i(t) - \theta_i^0 \), \((i = 1,2,\ldots,m)\) into Eqs. (5) and (6), now expressed in the functional forms

\[ f_i(\eta_1,\eta_2,\ldots,\eta_m,\dot{\eta}_1,\dot{\eta}_2,\ldots,\dot{\eta}_m,\ddot{\eta}_1,\ddot{\eta}_2,\ldots,\ddot{\eta}_m) = 0, \quad (i = 1,2,\ldots,m) \]  

(11)

and

\[ f_i(0,0,\ldots,0) = 0, \quad (i = 1,2,\ldots,m) \]  

(12)

respectively. Taylor series approximations of Eq. (11) about static equilibrium yield the linearized equations of motion for the electrodynamic system

\[ \sum_{j=1}^{m} (m_{ij} \ddot{\eta}_j + k_{ij} \eta_j) = 0, \quad (i = 1,2,\ldots,m) \]  

(13)

where

\[ m_{ij} = m_{ji} = \left( \frac{\partial f_i}{\partial \theta_j} \right)^0 = \sum_{k=1}^{n} m_k \frac{\partial r_k^0}{\partial \theta_i} \frac{\partial r_k^0}{\partial \theta_j} \]  

(14)

and

\[ k_{ij} = k_{ji} = \left( \frac{\partial f_i}{\partial \theta_j} \right)^0 = -\sum_{k=1}^{n-1} \sum_{l=k+1}^{n} \frac{k_{kl} q_l}{|r_l - r_i|^3} \left[ \frac{\partial r_l^0}{\partial \theta_i^2} (\delta_{ij} |r_k - r_i|^2 - 2 r_i^0) \frac{\partial (r_k^0 - r_i^0)}{\partial \theta_i} \frac{\partial (r_k^0 - r_i^0)}{\partial \theta_j} \right] \]

\[ + \left( \sum_{k=1}^{n} \frac{m_k g}{2L} \frac{\partial^2}{\partial \theta_j^2} |r_k - r_i|^2 \right) \delta_{ij} \]  

(15)
in which \( m_{ij} \) are elements of a positive definite and symmetric inertia matrix and \( k_{ij} \) are elements of a positive definite and symmetric stiffness matrix. These symmetry and definiteness properties imply that Eq. (13) admits normal mode behavior (Ref. 12). An associated set of \( m \) real modes of vibration are then governed by the eigenvalue problem

\[
\omega^{(k)^2} \sum_{j=1}^{m} m_{ij} \phi^{(k)}_j = \sum_{j=1}^{m} k_{ij} \phi^{(k)}_j , \quad (i,k = 1,2,\ldots,m)
\]

(16)

in which \( \phi^{(k)}_j , (j = 1,2,\ldots,m) \) denotes the \( k \)-th natural mode of vibration and \( \omega^{(k)} \) denotes the associated \( k \)-th natural frequency of oscillation. The natural modes of vibration are mutually orthogonal and can satisfy the orthonormality conditions

\[
\sum_{i=1}^{m} \sum_{j=1}^{m} \phi^{(r)}_i m_{ij} \phi^{(s)}_j = \delta_{rs} , \quad \sum_{i=1}^{m} \sum_{j=1}^{m} \phi^{(r)}_i k_{ij} \phi^{(s)}_j = \omega^{(r)^2} \delta_{rs} , \quad (r,s = 1,2,\ldots,m)
\]

(17)

**SOME TUTORIAL EXAMPLES**

This section gives tutorial examples of several electrodynamic systems.

**The electrostatic pendulum**

The position vectors in Eq. (1) are given by (see Fig. 2).

\[
\mathbf{\ell}_1 = -d_0 \mathbf{i}, \quad \mathbf{\ell}_2 = \mathbf{0}, \quad \mathbf{\ell}_3 = d(\cos \theta_1 \mathbf{i} + \sin \theta_1 \mathbf{j})
\]

Neglecting gravitational effects by letting \( g = 0 \) we obtain from Eq. (5) the equation of motion governing the electrostatic pendulum

![Figure 2. The electrostatic pendulum](image-url)
\[
m_3 \dddot{\theta}_1 + \frac{kq_1 q_3 d_0 d}{\left[ d_0^2 + d^2 + 2d_0 dc_1 \right]^{3/2}} s_1 = 0
\]

where \(c_1 = \cos \theta_1\) and \(s_1 = \sin \theta_1\). Let us now nondimensionalize the system by introducing the independent parameters

\[
p_1 = \frac{d_0}{d}, \quad p_2 = \frac{q_3}{q_1}, \quad p_3 = \frac{d^3 m_3}{kq_1^2}
\]

Introducing the independent parameters into the equation of motion yields

\[
0 = p_3 \dddot{\theta}_1 + \frac{p_1 p_2}{\left[ p_1^2 + 2p_1 c_1 + 1 \right]^{3/2}} s_1
\]

Clearly, the equilibrium position of the electrostatic pendulum satisfies

\[
p_1 p_2 s_1 \left[ p_1^2 + 2p_1 c_1 + 1 \right]^{3/2} = 0\implies \theta_1^0 = 0,
\]

that is that the static equilibrium angle is identically zero. Following the steps that lead to Eq. (13), the linearized electrostatic pendulum is governed by the harmonic equation of motion

\[
\ddot{\theta}_1 + \omega_p^2 \theta_1 = 0, \quad \omega_p^2 = \frac{p_1 p_2}{p_3 \left( p_1^2 + 2p_1 + 1 \right)^{3/2}}
\]

where \(\omega_p\) denotes the electrostatic pendulum frequency. Electrostatic pendulum frequencies are given in Table 1 for \(d = 0.0762m, p_2 = 1, V = kq_3/r, r = 0.018834m, k = 9.0 \cdot 10^9 \text{ F/m, and} m_3 = 2.7gr.\)

By applying a voltage \(V\) across each sphere of radius \(r\), and neglecting induction effects we obtain the natural frequencies given in Table 1.

<table>
<thead>
<tr>
<th>(p_1 = d_0/d)</th>
<th>20,000V</th>
<th>40,000V</th>
<th>60,000V</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.2358</td>
<td>2.4717</td>
<td>3.7076</td>
</tr>
<tr>
<td>0.6</td>
<td>1.3903</td>
<td>2.7806</td>
<td>4.1709</td>
</tr>
<tr>
<td>1.0</td>
<td>1.2843</td>
<td>2.5686</td>
<td>3.8529</td>
</tr>
<tr>
<td>1.4</td>
<td>1.1560</td>
<td>2.3120</td>
<td>3.4680</td>
</tr>
<tr>
<td>1.8</td>
<td>1.0402</td>
<td>2.0804</td>
<td>3.1206</td>
</tr>
</tbody>
</table>
The fixed charge electroscope

The position vectors in Eq. (1) are given by (see Fig. 3)

\[ r_1 = -d_0 \hat{i}, \quad r_2 = 0 \]
\[ r_3 = d(\cos \theta_1 \hat{i} + \sin \theta_1 \hat{j}), \quad r_4 = d(\cos \theta_2 \hat{i} - \sin \theta_2 \hat{j}) \]

Figure 3. The fixed charge electroscope

Once again letting \( g = 0 \), we obtain from Eq. (5) the equations of motion governing the fixed charge electroscope

\[
0 = m_3 \ddot{\theta}_1 + \frac{kq_1 q_3}{\left[ (d_0 + dc_1)^2 + (ds_1)^2 \right]^{3/2}} d_0 ds_1 \\
- \frac{kq_3 q_4}{\left[ d^2(c_1 - c_2)^2 + d^2(s_1 + s_2)^2 \right]^{3/2}} d^2(c_2 s_1 + s_2 c_1)
\]

\[
0 = m_4 \ddot{\theta}_2 + \frac{kq_1 q_4}{\left[ (d_0 + dc_2)^2 + (ds_2)^2 \right]^{3/2}} d_0 ds_2 \\
- \frac{kq_3 q_4}{\left[ d^2(c_1 - c_2)^2 + d^2(s_1 + s_2)^2 \right]^{3/2}} d^2(c_2 s_1 + s_2 c_1)
\]

where \( c_1 \equiv \cos \theta_1, c_2 \equiv \cos \theta_2, s_1 \equiv \sin \theta_2 \) and \( s_2 \equiv \sin \theta_2 \). Let us now nondimensionalize the system by introducing the independent parameters
\[ p_0 = d_0/d, \quad p_1 = m_3/m_4, \quad p_2 = q_2/q_1, \quad p_3 = q_3/q_1, \quad p_4 = q_4/q_1, \quad p_5 = d^3 m_4/k d_1^2 \]

Introducing the independent parameters into the equations of motion, yields

\[
0 = p_1 p_5 \ddot{\theta}_1 + \frac{p_0 p_3}{\left[ p_0^2 + 2p_0 c_1 + 1 \right]^{3/2}} s_1 - \frac{p_3 p_4}{\left[ 2 - 2c_1 c_2 + 2s_1 s_2 \right]^{3/2}} (c_2 s_1 + s_2 c_1)
\]

\[
0 = p_5 \ddot{\theta}_2 + \frac{p_0 p_4}{\left[ p_0^2 + 2p_0 c_2 + 1 \right]^{3/2}} s_2 - \frac{p_3 p_4}{\left[ 2 - 2c_1 c_2 + 2s_1 s_2 \right]^{3/2}} (c_2 s_1 + s_2 c_1)
\]

The static equilibrium position of the electroscope satisfies

\[
0 = \frac{p_0}{\left[ p_0^2 + 2p_0 c_1^0 + 1 \right]^{3/2}} s_1^0 - \frac{p_4}{\left[ 2 - 2c_1^0 c_2 + 2s_1^0 s_2 \right]^{3/2}} (c_2 s_1^0 + s_2 c_1^0)
\]

\[
0 = \frac{p_0}{\left[ p_0^2 + 2p_0 c_2^0 + 1 \right]^{3/2}} s_2^0 - \frac{p_3}{\left[ 2 - 2c_1^0 c_2 + 2s_1^0 s_2 \right]^{3/2}} (c_2 s_1^0 + s_2 c_1^0)
\]

The equilibrium positions are shown in Fig. 4.

\[ \theta_1 \]

1.642 rad

\[ \theta_2 \]

-1.532

-0.467

0.2

Figure 4. Equilibrium positions of the fixed charge electroscope
The linearized differential equations governing the motion of the fixed charge electroscope are now given by Eq. (13) in which

\[
\begin{align*}
&\quad \quad m_{11} = p_1 p_5, \quad m_{12} = m_{21} = 0, \quad m_{22} = p_5 \\
&k_{11} = \frac{p_0 p_3 c_1 [p_0^2 + 2p_0 c_1 + 1] + 3p_0^2 p_3 s_1^2}{[p_0^2 + 2p_0 c_1 + 1]^{5/2}} \\
&\quad - \frac{p_3 p_4 [(c_1 c_2 - s_1 s_2)(2 - 2c_1 c_2 + 2s_1 s_2) - 3(c_1 s_2 + c_2 s_1)^2]}{[2 - 2c_1 c_2 + 2s_1 s_2]^{5/2}} \\
&k_{12} = k_{21} = -\frac{p_3 p_4 [(c_1 c_2 - s_1 s_2)(2 - 2c_1 c_2 + 2s_1 s_2) - 3(c_1 s_2 + c_2 s_1)^2]}{[2 - 2c_1 c_2 + 2s_1 s_2]^{5/2}} \\
&k_{22} = \frac{p_0 p_3 c_2 [p_0^2 + 2p_0 c_2 + 1] + 3p_0^2 p_3 s_2^2}{[p_0^2 + 2p_0 c_1 + 1]^{5/2}} \\
&\quad - \frac{p_3 p_4 [(c_1 c_2 - s_1 s_2)(2 - 2c_1 c_2 + 2s_1 s_2) - 3(c_1 s_2 + c_2 s_1)^2]}{[2 - 2c_1 c_2 + 2s_1 s_2]^{5/2}}
\end{align*}
\]

where \(c_1 = \cos \theta_1, c_2 = \cos \theta_2, s_1 = \sin \theta_1 \) and \(s_2 = \sin \theta_2\). The natural modes of vibration and associated natural frequencies of oscillation are shown in Fig. 5. The natural modes of vibration shown are normalized such that \(\phi_2^{(1)} = \phi_2^{(2)} = 1\).

The electrostatic string

The position vectors in Eq. (1) are given by (see Fig. 6)

\[
\begin{align*}
&x_1 = 0, \quad x_2 = d\hat{i}, \quad x_k = x_{k-1} + d(\cos \theta_{k-2} \hat{i} + \sin \theta_{k-2} \hat{j}), \quad k = 3,4,\ldots,10
\end{align*}
\]

The static equilibrium position of the electrostatic spring is \(\theta_1^0 = \theta_2^0 = \ldots = \theta_8^0 = 0\). Linearization of the differential equations of motion lead to the natural modes of vibration and associated natural frequencies of oscillation shown in Fig. 7.
Figure 5. Natural modes and frequencies of the fixed charge electroscope

Figure 6. The electrostatic string
\[ \omega_1 = 0.038 \text{ (rad/sec)} \quad \omega_2 = 0.296 \quad \omega_3 = 0.655 \]

\[ q_r = 1 (r = 1, 2, \ldots, 10), \quad d = 1, \quad m = 1 \]

Figure 7. Lowest 3 natural modes and frequencies of the electrostatic string

A PHYSICAL ELECTROSCOPE EXPERIMENT

Consider the physical electroscope experiment shown in Fig. 8. The physical experiment and the previous analytical developments are related as follows:

(1) The point charges introduced in Eq. (3) represent conductive spheres. Induction effects causing uneven distributions of charge

Figure 8. Fixed charge electroscope experiment

over the surfaces of the spheres are neglected. Each conductive sphere is fabricated from a
0.018834m radius ping pong ball coated with Aquadag-E. The properties of Aquadag-E are given in Table 2.

<table>
<thead>
<tr>
<th>Table 2. Properties of Aquadag-E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
</tr>
<tr>
<td>Diluent</td>
</tr>
<tr>
<td>Freezing point</td>
</tr>
<tr>
<td>Maximum service temperature</td>
</tr>
</tbody>
</table>

(2) The effect of the grounded chamber is to introduce a null potential along the walls of the chamber in the plane of the motion of the spheres. The null potential is enforced by introducing massless image charges as shown in Fig. 9. In Eqs. (2) and (3), it follows that m = 2, n = 36 and \( q_{4l+3} = (-1)^{l+1} q_1, \ q_{4l+2} = (-1)^{l+1} q_2, \ q_{4l+1} = (-1)^{l+1} q_3, \ q_{4l} = (-1)^{l+1} q_4, \ (l = 1,2,\ldots,9) \). Throughout the experiment \( d = 0.0635 \text{m} \) (see fig. 9) and \( m_1 = m_2 = m_3 = m_4 = 2.7 \text{gr} \).

(3) The gravitational effect was eliminated in the experiment by suspending the two moving spheres from a common point directly above the central sphere and by using pinned rigid connections between the two moving spheres and the central sphere. Thus, we let \( g = 0 \) in Eq. (5).

(4) The four charged spheres are electrically connected to a common power supply providing a regulated voltage \( V = 20,000 \text{V} \). The sphere charges \( q_s \) \( (s = 1,2,3,4) \) are related to their sphere potentials \( V_r = V \) \( (r = 1,2,3,4) \) by

\[
V_r = \sum_{s=1}^{4} p_{rs} q_s, \quad q_s = \sum_{t=1}^{4} c_{st} V_t \quad (18a,b)
\]

in which \( p_{rs} \) \( (r,s = 1,2,3,4) \) are coefficients of potential and \( c_{st} \) \( (s,t = 1,2,3,4) \) are coefficients of capacitance. The coefficients \( p_{rs} \) and \( c_{st} \) depend on the position of the spheres. The electrical potential of each sphere in Eq. (18a) is given by
\[ \Delta x_1 = 0.9271m, \quad \Delta x_2 = 0.9271 - d \sin \theta_1, \quad \Delta x_3 = 0.9271 - d \sin \theta_2 \]
\[ \Delta y_1 = 0.5842m - d_0, \quad \Delta y_2 = d_0, \quad \Delta y_3 = 0.6070 - d \cos \theta_1, \quad \Delta y_4 = 0.6070 - d \cos \theta_2 \]

Figure 9. Enforcement of null potential by means of image charges
\[ V_r = \sum_{s=1, r \neq s}^{36} \frac{kq_s}{|\mathbf{r}_r - \mathbf{r}_s|} + \frac{kq_r}{r}, \quad (r = 1, 2, 3, 4) \] (19)

where \( r \) is the sphere radius. Substituting Eq. (19) into Eq. (18a), we obtain the matrix of coefficients of potential. Inverting this matrix and substituting the result into Eq. (18b), we obtain the computed charges as shown in Fig. 10. Note that the computed charges depend on position whereas the charges are assumed to be fixed in Eq. (5).

Indirect measurements of the positions of the spheres were obtained from the associated shadows created by high intensity light directed downward from the top of the electrodynamic chamber as shown in Fig. 11. The bottom surface of the chamber was lined with 0.2 in. spacing graph paper. The design of the chamber and related safety issues are described in Ref. 13.

The measured equilibrium angle and the computed equilibrium angle are compared in Fig. 12. As shown, the difference between the measured and computed equilibrium angles is on the order of the resolution of the grid paper. The effect of the grounded electrodynamic chamber is indicated in Table 3.

<table>
<thead>
<tr>
<th>Ratio</th>
<th>With no image</th>
<th>Including image</th>
<th>Measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80</td>
<td>59.89</td>
<td>57.67</td>
<td>57.25</td>
</tr>
<tr>
<td>1.00</td>
<td>60.00</td>
<td>57.91</td>
<td>57.75</td>
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<td>60.69</td>
<td>58.74</td>
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</tr>
<tr>
<td>1.40</td>
<td>61.71</td>
<td>59.93</td>
<td>59.9</td>
</tr>
<tr>
<td>1.60</td>
<td>62.91</td>
<td>61.34</td>
<td>61.1</td>
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<tr>
<td>1.80</td>
<td>64.19</td>
<td>62.92</td>
<td>62.9</td>
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<tr>
<td>2.00</td>
<td>65.51</td>
<td>64.63</td>
<td>64.5</td>
</tr>
</tbody>
</table>

The electroscope represents a two degree-of-freedom system that possesses two natural modes of vibration and two associated natural frequencies of oscillation as indicated in the tutorial example of the fixed-charge electroscope. The first mode of vibration is associated with the harmonic motion having the shape \((\theta_1, \theta_2) = (+1, -1)\). The second mode is associated with \((\theta_1, \theta_2) = \ldots\).
\( \theta_2 = (+1, +1) \). The natural modes and frequencies are computed from the eigenvalue problem, Eq. (16), with the image charges included in order to account for the effect of the grounded chamber. The measured natural frequencies and the computed natural frequencies are compared in Table 4. The 10%-20% error depicted in the results is attributed to the neglected nonlinearity wherein the charges are assumed to be fixed in time for purposes of linearization. Typical single mode excitations are shown in Figs. (13) and (14).

Table 4. Comparison of natural frequencies of oscillation (rad/sec)

<table>
<thead>
<tr>
<th>Ratio ((d_0/d))</th>
<th>Mode 1 Calculation</th>
<th>Mode 1 Measurement</th>
<th>Mode 2 Calculation</th>
<th>Mode 2 Measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>1.285</td>
<td>1.121</td>
<td>2.409</td>
<td>2.004</td>
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<tr>
<td>1.0</td>
<td>1.270</td>
<td>1.102</td>
<td>2.377</td>
<td>1.973</td>
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<tr>
<td>1.2</td>
<td>1.222</td>
<td>1.070</td>
<td>2.319</td>
<td>1.865</td>
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<tr>
<td>1.4</td>
<td>1.169</td>
<td>1.024</td>
<td>2.248</td>
<td>1.838</td>
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<tr>
<td>1.6</td>
<td>1.101</td>
<td>0.976</td>
<td>2.170</td>
<td>1.744</td>
</tr>
<tr>
<td>1.8</td>
<td>1.029</td>
<td>0.942</td>
<td>2.089</td>
<td>1.697</td>
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<tr>
<td>2.0</td>
<td>0.951</td>
<td>0.923</td>
<td>2.009</td>
<td>1.668</td>
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</table>

**CONCLUSION**

This paper analytically predicted and experimental confirmed the electrodynamic properties of interconnected charged particles. In the systems considered, the linearized vibration characteristics compared well with the nonlinear system in the sense that the linear effects dominated over the high-order nonlinear terms. Indeed equilibrium angles predictions compared with measurements with a 0.16° error and natural frequency predictions associated with the linearized system compared with measurements with a 10-20% linearization error. The dominance of the linearity is attractive since under proper conditions it implies for electrodynamic systems having more complex geometries and for electrodynamics with feedback control, that design strategies can readily be carried out on the basis of linear theories.
Figure 10. Computed charge of each sphere

Figure 11. Measurement system
Figure 12. Equilibrium angle versus length ratio

Figure 13. Mode No. 1 response (damping factor $\zeta = 0.007$, $d_0/d = 1.2$)
Figure 14. Mode No. 2 response (damping factor $\zeta = 0.07$, $d_0/d = 1.2$)
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REFERENCES


