Control of Non-Self-Adjoint Distributed-Parameter Systems

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Abstract. Systems involving viscous damping forces, circulatory forces, and aerodynamic forces are non-self-adjoint. A method capable of controlling non-self-adjoint distributed systems is the independent modal-space control method, whereby the problem of controlling a distributed-parameter system is reduced to that of controlling an infinite set of independent, complex, second-order ordinary differential equations. In the case of optimal control, one must solve independent, complex, scalar Riccati equations. The transient solution of the Riccati equations can be found with relative ease and the steady-state solution can be found in closed form. A numerical example demonstrates the effectiveness of the method.

Key Words. Control of non-self-adjoint systems, control of distributed-parameter systems, independent modal-space control, globally optimal control, complex scalar Riccati equations.

1. Introduction

A large class of distributed systems is characterized by a stiffness operator possessing the self-adjointness property. This implies a certain symmetry of the operator and the boundary conditions, which implies further that the system eigenfunctions are mutually orthogonal. The problem of control of self-adjoint distributed systems has been treated elsewhere (Ref. 1).

Damping tends to destroy the self-adjointness of distributed systems. Damping can arise from various sources such as viscous damping associated with shearing forces in fluids, aerodynamic effects, and internal hysteresis.

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where \( y(P, t) \) is the vector of displacements at any point \( P \) in the open spatial domain \( D \) of the system and \( f(P, t) \) represents a distributed control force. Moreover, \( m \) represents the mass matrix operator, \( c \) the damping matrix operator, and \( k \) the stiffness matrix differential operator of order 2\( p \). In addition, \( g \) denotes the gyroscopic matrix operator and \( h \) denotes the circulatory matrix operator. The displacement \( y(P, t) \) is subject to the boundary conditions

\[
B_i y(P, t) = 0, \quad P \in S, \quad i = 1, 2, \ldots, p, \tag{2}
\]

to be satisfied at every point \( P \) in \( S \) bounding \( D \), in which \( B_i \) represent matrix operators expressing homogeneous boundary conditions. The displacement \( y(P, t) \) and the distributed forces \( f(P, t) \) are elements of some Hilbert space \( Z \) with given norm \( \| \cdot \| \) and inner product \( \langle \cdot, \cdot \rangle \). The mass matrix operator \( m \) is self-adjoint; i.e., for any two admissible functions \( g_i(P, t) \) and \( g_j(P, t) \) in \( Z \) satisfying all the system boundary conditions (2), we have

\[
\langle g_n, mg_j \rangle = \langle g_n, mg_i \rangle. \tag{3}
\]

Likewise, the operators \( c \) and \( k \) are self-adjoint. The mass matrix is positive definite, the stiffness matrix operator is assumed to be positive semidefinite, and the damping matrix operator is positive semidefinite. The gyroscopic operator \( g \) is skew symmetric, i.e.,

\[
\langle g_n, gg_j \rangle = -\langle g_n, gg_i \rangle, \tag{4}
\]

where again \( g_i(P, t) \) and \( g_j(P, t) \) are admissible functions. The circulatory operator \( h \) is also skew symmetric.

It will prove convenient to rewrite the equations of motion (1) in state form. To this end, we introduce the state vector

\[
y(P, t) = [y^T(P, t), y^T(P, t)]^T
\]

in \( X = Z \times Z \) and write the state equations of motion

\[
M \ddot{y}(P, t) = L y(P, t) + F(P, t), \quad P \in D, \tag{5}
\]

where

\[
M = \begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} -(c + g) & -(k + h) \\ 0 & 1 \end{bmatrix}, \tag{6}
\]

\[
F(P, t) = [f^T(P, t), 0^T]^T. \tag{7}
\]

The state vector \( y(P, t) \) in Eq. (5) is an element of the Hilbert space \( X \) with inner product

\[
\langle \psi, \psi \rangle = \langle \ddot{y}, \ddot{y} \rangle + \langle y, y \rangle. \tag{8}
\]
\[ q_r(t) = \langle \psi_r(P), My(P, t) \rangle, \quad r = 1, 2, \ldots \]  \quad (15b)

Likewise, the distributed force in \( X \) can be expanded in the form

\[ F(P, t) = \sum_{r=1}^{\infty} M\dot{\phi}_r(P)Q_r(t), \quad r = 1, 2, \ldots \]  \quad (16a)

\[ Q_r(t) = \langle \psi_r(P), F(P, t) \rangle, \quad r = 1, 2, \ldots \]  \quad (16b)

Operating on the equations of motion (5) by \( \theta \), and considering Eqs. (15a) and (16a), we obtain the set of modal equations of motion

\[ \dot{q}_r(t) = \lambda_r q_r(t) + Q_r(t), \quad r = 1, 2, \ldots \]  \quad (17)

4. Coupled Control

Consider the distributed control force \( F(P, t) \) written in the linear feedback form

\[ F(P, t) = -G\psi(P, t), \]  \quad (18)

in which \( G \) denotes a linear control gain operator in \( X \). The control gain operator \( G \) is assumed to admit a discrete spectrum; hence, the distributed eigenvalue problems associated with the control gain operator \( G \) and its adjoint can be written as

\[ g_s M\dot{\phi}_s^*(P) = G\phi_s^*(P), \quad g_s M\dot{\psi}_s^*(P) = G^*\dot{\psi}_s^*(P), \quad r, s = 1, 2, \ldots \]  \quad (19)

where \( G^* \) is the adjoint of \( G \). The right and left eigenfunctions of \( G \), denoted by \( \phi_s^*(P) \) and \( \psi_s^*(P) \), are biorthogonal and can be normalized so that

\[ \langle \psi_s^*, M\phi_s^* \rangle = \delta_m, \quad \langle \psi_s^*, G\phi_s^* \rangle = g_s \delta_m \]  \quad (20)

where \( g_s \) are the control gain eigenvalues. Introducing the projection operator \( \langle \psi_s^*, M \rangle M\phi_s^* \) of \( G \), we can resolve \( G \) as follows:

\[ G = \sum_{r=1}^{\infty} g_r \langle \psi_r^*, M \rangle M\phi_r^*. \]  \quad (21)

Moreover, we can transform the control law (18) into the modal space. Operating on Eq. (18) by \( \theta \), we obtain the modal control law

\[ Q_r(t) = -\sum_{s=1}^{\infty} g_s q_s(t), \]  \quad (22a)

\[ g_s = \langle \psi_s^*, G\phi_s^* \rangle, \]  \quad (22b)
so that, inserting Eqs. (27) into Eqs. (17), we obtain the closed-loop modal equations

\[ \dot{q}_r(t) + (g_r - \lambda_r) q_r(t) = 0, \quad r = 1, 2, \ldots \]  

Equations (28) are now both internally and externally decoupled. Next, we consider three properties pertinent to the independent modal-space control method.

**Property 5.1.** The eigenfunctions associated with the open-loop system and the closed-loop system are identical if and only if the control is independent modal-space control.

For independent modal-space control, it is clear from Eqs. (28) that the closed-loop equations are decoupled. Therefore, the closed-loop eigenfunctions are identical to the open-loop eigenfunctions. The differential eigenvalue problem associated with the closed-loop system, in which the closed-loop eigenfunctions are identical to the open-loop eigenfunctions, has the form

\[ s_r M \phi_r(P) = (L - G) \phi_r(P), \quad r = 1, 2, \ldots \]  

Adding Eq. (9a) to Eq. (29), we obtain the differential eigenvalue problem associated with the gain operator \( G \), in which the closed-loop eigenfunctions and the control eigenfunctions are identical.

**Property 5.2.** The closed-loop eigenvalues can be arbitrarily placed in the complex plane.

It follows from Eqs. (28) that the control gain eigenvalues corresponding to arbitrarily placed closed-loop poles \( s_r \) are given by

\[ g_r = \lambda_r - s_r, \quad r = 1, 2, \ldots \]  

**Property 5.3.** The independent modal-space control, in which the performance functional

\[ J = \sum_{r=1}^{\infty} h_r (q_r(t_f) - \bar{q}_r)(\bar{q}_r(t_f) - \bar{q}_r) + \int_0^{t_f} (q_r \dot{q}_r + R_r Q_r \dot{Q}_r) \, dt \]  

is minimized, is unique and globally optimal.

In (31), \( \bar{q}_r \) is a reference value to which \( q_r \) is to be driven at the final time \( t_f \) and \( R_r, h_r \) are real, positive modal weights. Note that each term in the summation represents a real quadratic functional corresponding to a mode of vibration. It is well known that the linear optimal solution is unique (Ref. 9). Moreover, considering Eqs. (17) and recognizing that the control forces \( Q_r \) are unconstrained, we can replace the minimization of \( J \) as given
where primes denote differentiation with respect to $t$, and in which
\[
M_r = \begin{bmatrix}
-a_r & -R_r^{-1} \\
-1 & a_r
\end{bmatrix}, \quad r = 1, 2, \ldots, \tag{39}
\]
are real matrices. The solution to Eqs. (38) can be obtained with relative ease.

6. Control Implementation

In Section 4, it was demonstrated how the independent modal-space control method can generate distributed controls $F(P, t)$ for distributed-parameter systems. In practice, infinite-dimensional controls may not be possible, in which case the interest lies in implementing independent modal-space control with a finite number of actuators and sensors. The problem reduces to how to approximate distributed controls and measurements by discrete controls and measurements.

Let us write the actual control force as a projection of the computed distributed control force
\[
F(P, t) = SF^*(P, t), \tag{40}
\]
where $S$ denotes the projection operator under consideration and $F^*(P, t)$ is the computed distributed control force. Similarly, the computed distributed displacement $w^*(P, t)$ is a projection of the actual displacement $w(P, t)$ having the form
\[
w^*(P, t) = Tw(P, t), \tag{41}
\]
where $T$ represents the projection operator. Given the projections $S$ and $T$, the closed-loop eigenvalues can be found. Indeed, substituting Eq. (40) into Eqs. (16b), we obtain
\[
Q_r = \langle \psi_r, SF^* \rangle = \sum_{s=1}^\infty S_r Q^*_r, \quad r = 1, 2, \ldots, \tag{42}
\]
where
\[
S_r = \langle \psi_r, SM_\phi \rangle, \quad r, s = 1, 2, \ldots, \tag{43}
\]
and $Q^*_r, s = 1, 2, \ldots, $ are computed modal controls. Moreover, substituting Eq. (41) into Eqs. (15b), we obtain
\[
q^*_s = \langle \psi_s, MT \psi \rangle = \sum_{t=1}^\infty T_s q_t, \quad s, t = 1, 2, \ldots, \tag{44}
\]
Independent modal-space control requires the modal coordinates. They can be obtained by means of Eqs. (15b). In a process similar to that used in the finite-element method, the distributed state can be represented by linear combination of interpolation functions multiplied by nodal coordinates (Ref. 11),

$$\psi(P, t) = F(P)\psi(t),$$

in which $F(P)$ represents a rectangular matrix of interpolation functions and $\psi(t)$ is a vector of nodal coordinates. Substituting Eq. (53) into Eqs. (15b), we obtain the modal coordinates from the nodal coordinates as follows:

$$q_r(t) = F^T_r\psi(t), \quad F_r = \int_D F^T(P)M\psi_r(P)\,dD, \quad r = 1, 2, \ldots$$

7. Numerical Example

As an example, we consider a simply supported uniform shaft whirling about its axis at the constant angular velocity $\Omega = 100$. The shaft extends 10 units in the $x$ direction and undergoes displacements $u_x(z, t)$ and $u_y(z, t)$ in the directions of the body-fixed axes $x$ and $y$, respectively, as shown in Fig. 1. The distributed displacement vector in Eq. (1) has the form

$$y(P, t) = [u_x(z, t), u_y(z, t)]^T.$$ 

Moreover, the specific quantities in Eq. (1) are given by (Ref. 11)

$$m = \begin{bmatrix} m_0 & 0 \\ 0 & m_0 \end{bmatrix}, \quad c = \begin{bmatrix} c_0 + h_0 & 0 \\ 0 & c_0 + h_0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 & -2m_0\Omega \\ 2m_0\Omega & 0 \end{bmatrix}.$$  

Fig. 1. Uniform shaft whirling at constant angular velocity $\Omega$. 

$$u(z, t) = \begin{bmatrix} u_x(z, t) \\ u_y(z, t) \end{bmatrix}.$$
control forces. However, practical considerations often preclude the use of distributed control forces, so that the analysis is extended so as to include discrete-point controls. Consistent with this, the modal coordinates needed for feedback are extracted from measurements taken at discrete points. As a numerical example, a uniform shaft rotating about its axis with constant angular velocity was considered. For a given angular velocity, the lowest two modes are unstable. These two modes were controlled successfully using two pairs of discrete-point actuators.

References